JOHANNES KEPLER
UNIVERSITY LINZ

Submitted by
Dipl.-Ing.
Benjamin Eichinger

Submitted at
Institute for Analysis

Supervisor and
First Examiner
Univ.Doz.Dr.
Petro Yudytskiy

Second Examiner
Assoc.Prof.Doc.Dr.
Jacob S. Christiansen

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Periodic GMP matrices and asymptotics of extremal polynomials for Chebyshev and Ahlfors problems in the complex plane

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JOHANNES KEPLER
UNIVERSITY LINZ
Altenbergerstraße 69
4040 Linz, Österreich
www.jku.at
DVR 0093696

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Abstract

This thesis deals with two main topics. The first part is devoted to the spectral theory of periodic GMP matrices and the second part deals with asymptotics of extremal polynomials for Ahlfors and Chebyshev problems in the complex plane.

It is well known that orthogonal polynomials associated to a compactly supported real measure obey a three-term recurrence relation with uniformly bounded coefficient sequences. This recurrence relation naturally induces an operator on $\ell^2(\mathbb{Z}_{\geq 0})$. These operators are called one-sided Jacobi matrices. Given a finite system of intervals, E, the finite gap class, $J(E)$, of Jacobi matrices is the set of two-sided Jacobi matrices which have purely absolutely continuous spectrum, supported on the given set and which are almost periodic. It is a property of the set whether the elements in the finite gap class are even periodic. In this case there exists a polynomial, T, (the normalized Chebyshev polynomial of degree n associated to the set, where n is the length of the period) such that $E = T^{-1}([-1, 1])$. This polynomial can be used to given an algebraic description of $J(E)$ by means of the coefficients of the Jacobi matrices. Moreover, the finite gap class can also be characterized by a certain operator identity, called the magic formula. Due to Akhiezer and others [6] it is known that the theory of finite gap Jacobi matrices is intimately related to function theory on hyperelliptic Riemann surfaces. We survey this relation in the preliminary chapter. In particular, we present several equivalent conditions on the set E, which are responsible for the fact that $J(E)$ consists of periodic operators. These conditions are the motivation for the definition of periodic GMP matrices. They form a certain **G**eneralization of matrices related to the strong **M**oment **P**roblem. This class allows us to give a parametrization of *almost periodic* finite gap Jacobi matrices by *periodic* GMP matrices. Moreover, due to their structural similarity we can carry over numerous results from the direct and inverse spectral theory of periodic Jacobi matrices to the class of periodic GMP matrices. In particular, we prove an analog of the magic formula for this new class.

The nth Chebyshev polynomial associated to a compact subset, K, of $\mathbb{C}$ is that monic polynomial of degree n, which minimizes the sup-norm, $\|\cdot\|_K$, on the given set. Thiran and Detaille [62] gave an explicit formula for the asymptotics of the sup-norm of the Chebyshev polynomials associated to a circular arc. We give the so-called Szegő-Widom asymptotics for this domain, i.e., explicit expressions for the asymptotics of the corresponding extremal polynomials. This notion was introduced by Christiansen, Simon and Zinchenko who proved this kind of asymptotics for the Chebyshev polynomials associated to finite systems of intervals [16]. Moreover, we solve a similar problem with respect to the upper envelope of a family of polynomials uniformly bounded on this arc. That is, we

give explicit formulas for the asymptotics of the error of approximation as well as of the extremal functions. Our computations show that in the proper normalization the limit of the upper envelope represents the diagonal of a reproducing kernel of a certain Hilbert space of analytic functions. Due to Garabedian, the analytic capacity in an arbitrary domain is the diagonal of the corresponding Szegő kernel.

Motivated by the result of Garabedian, classical results of Widom and the affirmative answer for circular arcs, we raised a conjecture that the asymptotics for Chebyshev polynomials in a complex domain can be given in terms of the reproducing kernels of a suitable Hilbert space of analytic functions in this domain. In the last section, in order to support this conjecture we study asymptotics for Ahlfors extremal polynomials in the complement to a system of intervals on $\mathbb{R}$, arcs on $\mathbb{T}$, and its continuous counterpart.

Zusammenfassung

Diese Dissertation kann in zwei Hauptthemenfelder aufgeteilt werden. Der erste Teil befasst sich mit der Theorie periodischer GMP Matrizen und der zweite Teil mit der Asymptotik von extremalen Polynomen bzgl. Problemstellungen von Tschebyscheff und Ahlfors in der komplexen Ebene.

Es ist bekannt, dass orthogonale Polynome bezüglich eines kompakt getragenen reellen Maßes eine dreistufige Rekursionsvorschrift erfüllen. Diese Beziehung induziert auf natürliche Weise Operatoren auf $\ell^2(\mathbb{Z}_{\geq 0})$. Diese Operatoren werden Jacobi Matrizen genannt. Für eine gegebene Vereinigung von Intervallen E, bezeichnet die "finite-gap" Klasse von Jacobi Matrizen, jene Menge von Jacobi Matrizen, deren Spektralmaß die gegebene Menge als Träger hat und rein absolut stetig ist und welche fast-periodisch sind. Diese Klasse wird auch als isospektraler Torus bezeichnet. Wir werden im Einleitungsteil der Dissertation erläutern, dass die Bezeichnung Torus tatsächlich gerechtfertigt ist. Es ist eine Eigenschaft der Menge E, ob die Elemente des isospektralen Torus nicht nur fast-periodisch, sondern sogar periodisch sind. Im zweiten Fall existiert ein Polynom T, sodass $E = T^{-1}([-1, 1])$ (das normalisierte Tschebyscheff Polynom vom Grad n bezüglich der Menge E, wobei n die Länge der Periode ist). Vermöge dieses Polynoms ist es möglich eine algebraische Beschreibung des isospektralen Torus anhand der Koeffizientenfolgen anzugeben. Außerdem kann der isospektrale Torus durch eine gewisse Operatorenidentität, die sogenannte magische Formel, charakterisiert werden. Akhiezer [6] erkannte den Zusammenhang zwischen der Theorie der finite gap Klasse und der Theorie hyperelliptischer Riemannscher Flächen. Wir behandeln diese Zusammenhänge im Einleitungskapitel. Im Speziellen präsentieren wir einige äquivalente Bedingungen an die Menge E, die für die Periodizität der assoziierten Jacobi Matrizen verantwortlich sind. Diese Bedingungen dienen als Motivation für die Definition von periodischen GMP Matrizen. Diese bilden eine **G**eneralisierung von Matrizen, welche mit dem **S**tarken **M**omentenproblem verbunden sind. Diese Klasse erlaubt uns eine Parametrisierung der *fast-periodischen* finite gap Jacobi Matrizen mithilfe von *periodischen* GMP Matrizen anzugeben. Weiters, dank struktureller Ähnlichkeiten, können wir viele Resultate der direkten und inversen Spektraltheorie von Jacobi Matrizen auf die Klasse der GMP Matrizen übertragen. Im Speziellen beweisen wir eine Analogon der magischen Formel für GMP Matrizen.

Das n-te Tschebyscheff Polynom, assoziiert mit einer kompakten Teilmenge K der komplexen Ebene, ist jenes monische Polynom vom Grad n, welches in dieser Klasse die Supremumsnorm $\|\cdot\|_K$ minimiert. Thiran und Detaille [62] haben eine explizite Formel für die Asymptotik der Supremumsnorm der Tschebyscheff Polynome, assoziiert mit einem Kreisbogen des Einheitskreises, angegeben. Wir

finden explizite Ausdrücke für die Asymptotik der entsprechenden extremalen Funktionen T_n. Christiansen, Simon und Zinchenko haben ein entsprechendes Resultat für Vereinigungen von Intervallen bewiesen [16] und dafür die Bezeichnung Szegő-Widom Asymptotik eingeführt. Weiters lösen wir ein verwandtes Problem für die obere Einhüllende von Polynomen welche gleichmäßig auf dem gegebenen Kreisbogen beschränkt sind. Das heißt wir geben explizite Formeln für die Asymptotik des extremalen Wertes und der extremalen Funktionen an. Unsere Berechnungen ergeben, dass der Grenzwert des extremalen Wertes als Funktion in der Komplementärmenge durch die Diagonale eines reproduzierenden Kernes eines Hilbertraumes analytischer Funktionen angegeben werden kann. Garabedian hat gezeigt, dass die analytische Kapazität in einem Gebiet durch die Diagonale des entsprechenden Szegő Kernes repräsentiert wird.

Motiviert durch Garabedians Resultat, klassischer Resultate Widoms und unseres bestätigenden Beispiels des Kreisbogens, haben wir die Vermutung geäußert, dass die Asymptotik von Tschebyscheff Polynomen durch die Diagonale eines reproduzierenden Kernes eines geeigneten Hilbertraumes analytischer Funktionen beschrieben werden kann. Um diese Vermutung zu unterstützen behandeln wir im letzten Kapitel die Asymptotik von Ahlfors Polynomen im Komplement von der Vereinigung reeller Intervalle und Kreisbögen.

Structure

The preliminary chapter should serve as a condensed survey on selected topics of the theory of Chebyshev polynomials associated to subsets of the real line and orthogonal polynomials for compactly supported real measures. Particularly, we try to present the connections between these to theories. The first part is mainly based on the survey article [58] and the second part presents the construction developed in [59], but follows the presentation in [66].

The results presented in Chapter 2 are based on three papers. Section 2.1 presents the result of [21].

The following two sections are devoted to our results on asymptotics for polynomial uniform approximation problems. In Section 2.2.1 we present the main results of [22] and Section 2.2.2 is based on the results of [24].

In Chapter 3, 4.1 and 4.2 we successively give the proofs of the results presented in Chapter 2.

Contents

Chapter 1

Preliminaries

1.1 Chebyshev polynomials

Let $K \subset \mathbb{C}$ be compact. The nth Chebyshev polynomial, T_n, associated with the set K is the monic polynomial of degree n that minimizes the sup-norm

$$\|f\|_K := \sup_{z\in K} |f(z)|,$$

in this class. Let $\mathcal{P}_n$ denote the class of polynomials of degree at most n. First, we would like to clarify existence and uniqueness.

Lemma 1.1. *Let K be a compact subset of $\mathbb{C}$ containing at least $n+1$ points. Then there exists a unique Chebyshev polynomial, T_n, of degree n associated with the set K.*

Proof. By compactness existence is guaranteed. By a variational argument we see that there exist at least $n+1$ points on K at which $|T_n(z)| = \|T_n\|_K$. Assume that there are only n points $z_1, \dots, z_n \in K$ at which $|T_n(z)|$ is maximal. Then there exists a polynomial Q_{n-1} such that $Q_{n-1}(z_j) = T_n(z_j)$. Note that this system of equations can be written as

$$\begin{bmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ \vdots & & \vdots & \\ 1 & z_n & \dots & z_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} T_n(z_1) \\ \vdots \\ T_n(z_n) \end{bmatrix}.$$

The matrix on the left-hand side is the well known Vandermonde matrix, $\mathcal{V}$, with $\det \mathcal{V} = \prod_{i<j}(z_i - z_j)$. Hence, we can choose $\epsilon > 0$ small enough such that $\|T_n - \epsilon Q_{n-1}\|_K < \|T_n\|_K$, which is a contradiction. Assume that there are 2 different Chebyshev polynomials, so is their average $(T_n + Q_n)/2$. At the $n+1$ extremal points of the average we have $T_n = Q_n$. Since the difference $T_n - Q_n$ is a polynomial of degree $n-1$ with $n+1$ zeros, it must vanish identically. □

1.1.1 Chebyshev polynomials associated to a finite system of intervals

We would like to start with Chebyshev polynomials associated to subsets of the real line as the shape of these polynomials is particularly accessible. We will see later that for sets $E_\infty = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{i=1}^{\infty} (\mathbf{a}_i, \mathbf{b}_i)$ there exists always a finite system of intervals, E, such that the Chebyshev polynomials associated to E and to E_∞ coincide. Therefore we will assume throughout this section that

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{i=1}^{g} (\mathbf{a}_i, \mathbf{b}_i), \quad g \in \mathbb{N}. \tag{1.1}$$

Note that if $T_n(z)$ is the Chebyshev polynomial for E, so is $\overline{T_n(\overline{z})}$. Hence, uniqueness implies that T_n is real.

Definition 1.2. For a polynomial P, let $M_P := \{x \in E : |P(x)| = \|P\|_E\}$. A set of alternation is a maximal subset of M_P, $x_0 < x_1 < \cdots < x_k$, such that P assumes the value $\|P\|_E$ with alternating sign.

Point (v) of the following theorem is known as the Chebyshev alternation theorem, although it is not mentioned in any of his works. Our proof follows a method of Markov [44].

Theorem 1.3. *Let E be of the form (1.1). Then there exists a unique Chebyshev polynomial, T_n, which satisfies:*

(i) $\{x : T_n(x) = 0\} \subset (\mathbf{b}_0, \mathbf{a}_0)$,

(ii) T_n *has only simple zeros,*

(iii) $(-1)^n T_n(\mathbf{b}_0) = T_n(\mathbf{a}_0) = \|T_n\|_E$,

(iv) Between two successive zeros of T_n, x_1, x_2, there is always a point $x \in (x_1, x_2) \cap E$ such that $|T_n(x)| = \|T_n\|_E$.

(v) The set of alternation points of T_n consists of $n+1$ points. Moreover, this property characterizes T_n uniquely.

Proof. (i) Let us first prove that T_n has no complex zeros. If $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is a zero of T_n, so is $\overline{z}_0$. We can choose $\epsilon > 0$ small enough such that

$$\tilde{T}_n(z) := T_n(z) \left(1 - \epsilon \frac{1}{(z - z_0)(z - \overline{z}_0)} \right)$$

satisfies $\|\tilde{T}_n\|_E < \|T_n\|_E$. By assumption, $\tilde{T}_n$ is a monic polynomial of degree n and therefore this is a contradiction. If $x_0 < \mathbf{b}_0$ is a zero, the polynomial

$$\tilde{T}_n(z) := T_n(z) \left(1 - \epsilon \frac{1}{(z - x_0)} \right)$$

leads to a contradiction in the same way. Similarly we show that there can't be a zero which is larger than $\mathbf{a}_0$. For (ii) let us assume WLOG that $x_0 \in E$. We use the correction term

$$\tilde{T}_n(z) := T_n(z) \left(1 - \epsilon \frac{1}{(z - x_0)^2} \right).$$

In a vicinity $\mathcal{U}_\delta(x_0) \subset E$ around x_0 we have $\tilde{T}_n = T_n - \epsilon Q_{n-2}$. Since T_n vanishes in x_0, we can choose first δ and then ϵ small enough such that $\sup_{x \in \mathcal{U}_\delta(x_0)} |\tilde{T}_n(x)| < \|T_n\|_E$. Outside of $\mathcal{U}_\delta(x_0)$ we clearly can find an appropriate ϵ such that we again obtain that $\|\tilde{T}_n\|_E < \|T_n\|$. If $x_0 \notin E$ then the same correction term with suitable ϵ leads to a contradiction. (iii) Since T_n has only simple real zeros, the zeros and critical points of T_n are interlacing. Let $x_0 = \max\{x : T_n(x) = 0\}$. If $T_n(\mathbf{a}_0) < \|T_n\|_E$, then this holds for all points in $[x_0, \mathbf{a}_0]$. For $\epsilon > 0$ small enough we define

$$\tilde{T}_n(z) := T_n(z)\left(1 + \epsilon \frac{1}{(z - x_0)}\right).$$

Note that on the interval $[x_0, \mathbf{a}_0]$ the expression in brackets is larger than 1. But on this interval we are allowed to make T_n slightly larger and $|\tilde{T}_n(x)| < \|T_n\|_E$ will still hold. Hence, with the right choice of ϵ we see that $\|\tilde{T}_n\|_E < \|T_n\|_E$.
In a similar way one can show (iv).
The first part of (v) follows by counting. For the second part, assume that there exists a monic P_n with $n+1$ alternating points such that $\|T_n\|_E < \|P_n\|_E$. Then at these $n+1$ points, $Q_{n-1} = P_n - T_n$, which is a polynomial of degree $n-1$, has the same sign as P_n. Thus, it has n zeros and therefore it vanishes identically. □

Riemann surfaces, conformal mappings and potential theory

We define the *n-extension* of the set E by

$$E_n := T_n^{-1}([-\|T_n\|_E, \|T_n\|_E]) = \{z \in \mathbb{C} : \; T_n(z) \in [-\|T_n\|_E, \|T_n\|_E]\}.$$

Note that Theorem 1.3 implies that $E_n \subset [\mathbf{b}_0, \mathbf{a}_0]$. Let us introduce $\mathbf{u}_i, \mathbf{v}_i \in [\mathbf{b}_0, \mathbf{a}_0] \setminus \mathring{E}$ with $\mathbf{u}_i \le \mathbf{v}_i$ for $1 \le i \le \tilde{g}$, where $\tilde{g} \in \mathbb{N}$ and $\mathring{E}$ denotes the interior of E such that

$$E_n = E \cup \bigcup_{i=1}^{\tilde{g}} [\mathbf{u}_i, \mathbf{v}_i].$$

Proposition 1.4. *We can choose $\mathbf{a}_i \le \mathbf{u}_i \le \mathbf{v}_i \le \mathbf{b}_i$. In particular $\tilde{g} = g$ and only the following cases are possible*

(i) The gap is closed. That is $\mathbf{a}_i = \mathbf{u}_i$ and $\mathbf{b}_i = \mathbf{v}_i$,

(ii) E is extended on one edge. That is, either $\mathbf{a}_i = \mathbf{u}_i$ and $\mathbf{v}_i < \mathbf{b}_i$ or $\mathbf{a}_i < \mathbf{u}_i$ and $\mathbf{v}_i = \mathbf{b}_i$,

(iii) The gap remains unchanged. In this case we set $\mathbf{a}_i = \mathbf{u}_i = \mathbf{v}_i$,

(iv) An internal interval is added. That is, $\mathbf{a}_i < \mathbf{u}_i < \mathbf{v}_i < \mathbf{b}_i$.

Proof. The alternation theorem implies that if for a critical point c we have $|T_n(c)| = \|T_n\|_E$, then $c \in E$. If $|T_n(c)| > \|T_n\|_E$, take the smallest interval $[x_1, x_2]$ such that $c \in [x_1, x_2]$ and such that $|T_n(x_i)| = \|T_n\|_E$. Then at least one of two endpoints must be on the original set E. This shows that only the mentioned cases are possible. □

It is convenient to write

$$E_n = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{i=1}^{m} (\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i), \quad m \leq 2g \wedge m \leq n-1.$$

We are now able to explain why for fixed n it suffices to consider only finite gap sets. Even if g would be infinite, the Chebyshev polynomial of E and E_n would be the same. That is, almost all gaps would be closed.

In the following it will be more convenient to use a different normalization for the Chebyshev polynomial. Let $P_{n,\infty} = \frac{T_n}{\|T_n\|}$ be the degree n polynomial that is bounded by one in modulus on E and has maximal leading coefficient in this class. Moreover, let us define the function

$$R_{E_n}(z) = (z - \mathbf{a}_0)(z - \mathbf{b}_0) \prod_{i=1}^{m} (z - \boldsymbol{\alpha}_i)(z - \boldsymbol{\beta}_i).$$

Then

$$1 = P_{n,\infty}(z)^2 - R_{E_n}(z) Q_{n-m-1}(z)^2, \tag{1.2}$$

where $Q_{n-m-1}(z)$ is a polynomial having zeros at the double zeros of the difference $1 - P_{n,\infty}(z)^2$.

Already due to Akhiezer is the idea of interpreting functional equations of the type (1.2), arising for extremal problems, as factorization identities on suitable Riemann surfaces; see e.g. [4, 5]. More references can be found in [58]. Introducing the function

$$\mathcal{E}(z) = P_{n,\infty}(z) - \sqrt{R_{E_n}(z)} Q_{n-m-1}(z), \quad \mathcal{E}^\tau(z) = P_{n,\infty}(z) + \sqrt{R_{E_n}(z)} Q_{n-m-1}(z), \tag{1.3}$$

where the square root is chosen such that $\sqrt{R_{E_n}(z)} > 0$ for $z > \mathbf{a}_0$, (1.2) reads

$$1 = \mathcal{E}(z)\mathcal{E}^\tau(z). \tag{1.4}$$

This suggests to interpret this as a factorization on the Riemann surface

$$\mathcal{R} = \{p = (z, w) : w^2 = R_{E_n}(z)\} \cup \{\pm\infty\}. \tag{1.5}$$

Let $\mathcal{R}_\pm$ be two copies of $\Omega_n = \overline{\mathbb{C}} \setminus E_n$ glued together along E_n in the usual way; see Section A.1 or [46, Chapter IIIa]. Let us fix the upper sheet $\mathcal{R}_+$ by demanding that $w > 0$ for $z > \mathbf{a}_0$. On $\mathcal{R}$ we define the involution $\tau p = (z, -w)$. In abuse of notation, we use $\mathcal{E}$ also for the rational function on $\mathcal{R}$ defined by

$$\mathcal{E}(p) = P_{n,\infty}(z) - w Q_{n-m-1}(z), \quad p = (z, w).$$

Hence, (1.4) is a factorization by means of this function with $\mathcal{E}^\tau(p) = \mathcal{E}(\tau p)$. Let us find the divisor of $\mathcal{E}$. By the choice of the square root and (1.4), $\mathcal{E}$ has a zero of multiplicity n at ∞_+ and a pole of the same degree at ∞_-. The strategy is the following: To our given set E we associate a family of Riemann surfaces parametrized by the unknown parameters $\{\mathbf{u}_i, \mathbf{v}_i\}$. The existence of the rational function $\mathcal{E}$ on $\mathcal{R}$ with divisor $n(\infty_+ - \infty_-)$ will finally determine these parameters.

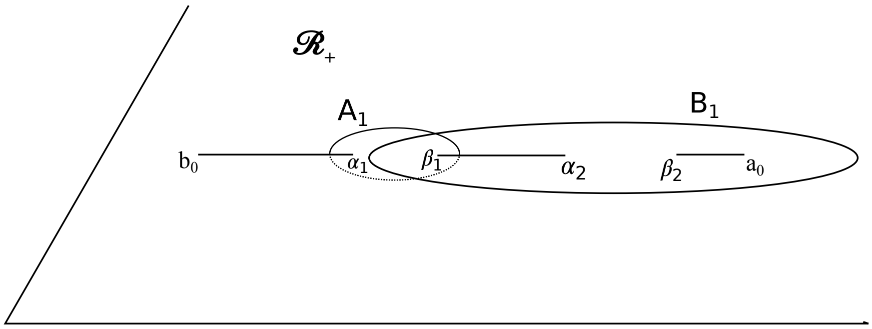


Figure 1.1: A and B cuts

Let $E_n^i = [\boldsymbol{\beta}_i, \mathbf{a}_0] \cap E_n$. Moreover, let A_i, B_i be a canonical basis of cycles on $\mathcal{R}$. To be more precise, let A_i be contours encircling the gap $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ clockwise and B_i be a contour on $\mathcal{R}_+$ that starts at ∞_+ goes into the upper half-plane, passes through the gap $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ and returns on the lower half-plane such that E_n^i is in its interior; cf. Figure 1.1.

By $\{\mathrm{d}j_1^l\}_{l=1}^g$ we denote the basis of Abelian differentials of the first kind normalized such that the matrix of periods $\Pi^A = \{\int_{A_k} \mathrm{d}j_1^l\}_{k,l=1}^m$ is the identity matrix. They are of the form

$$\mathrm{d}j_1^l(z,w) = C_l \frac{\prod_{i \neq l}(z - \mathbf{x}_i)}{w} \mathrm{d}z, \tag{1.6}$$

where

$$\mathbf{x}_i \in (\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i), \quad C_l^{-1} = 2 \int_{\boldsymbol{\alpha}_l}^{\beta_l} \frac{\prod_{i \neq l}(z - \mathbf{x}_i)}{w} \mathrm{d}z < 0.$$

The following lemma gives the first of several connections between the theory of compact Riemann surfaces and potential theory. For a reference on potential theory see e.g. [53]. Let $\omega(z, I; \Omega)$ denote the harmonic measure of the set I evaluated at the point z of the domain Ω.

Lemma 1.5. *Let*

$$j_1^l(p) = \int_{\mathbf{b}_0}^p \mathrm{d}j_1^l.$$

Then

$$2 \operatorname{Re} j_1^l(z,w) = \omega(z, E_n^l; \Omega_n), \quad (z,w) \in \mathcal{R}_+. \tag{1.7}$$

Proof. On Ω_n, $j_1^l(z,w)$ acts like a Schwarz-Christoffel transformation and we see that it maps the lower half-plane onto a rectangular comb like it is shown in Figure 1.2. $\operatorname{Re} j_1^l$ can then be extended as a single-valued harmonic function to Ω_n. Since the line $\operatorname{Re} j_1^l = 0$ and $\operatorname{Re} j_1^l = \frac{1}{2}$ corresponds to $E_n \setminus E_n^l$ and E_n^l, respectively, we obtain (1.7). □

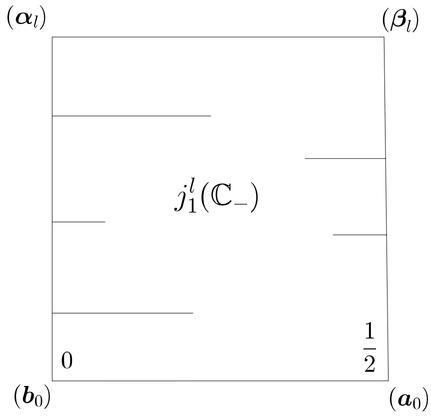


Figure 1.2: Comb related to j_1^l

Lemma 1.6. *Let E_n be the n-extension of a set E. Then*

$$n\omega(\infty, E_n^l; \Omega_n) \in \mathbb{Z} \quad \textit{for all} \quad 1 \le l \le m.$$

Proof. Abel' s theorem, provides a criterion for existence of a rational function for a given divisor, cf. Theorem A.1 In particular, this implies that a necessary condition for the existence of a rational function with divisor $n(\infty_+ - \infty_-)$ is that

$$\mathbb{Z} \ni n(\operatorname{Re} j_1^l(\infty_+) - \operatorname{Re} j_1^l(\infty_-)) = 2n \operatorname{Re} j_1^l(\infty_+) = n\omega(\infty, E_n^l; \Omega_n).$$

□

The corresponding extremal function can be recovered by means of Abelian integrals of the third kind. Let $\mathrm{d}j_3(p) = \mathrm{d}j_3(p; \infty_+, \infty_-)$ be the Abelian differentials of the third kind with $\operatorname{Res}_{\infty_\pm} \mathrm{d}j_3 = \pm 1$ normalized by the condition that its A-periods vanish. Due to this normalization, it is of the form

$$\mathrm{d}j_3(z, w) = -\frac{\prod_{i=1}^m (z - \mathbf{d}_i)}{w} \mathrm{d}z, \quad \mathbf{d}_i \in (\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i),$$

where the points $\mathbf{d}_i$ are determined by

$$\int_{\boldsymbol{\alpha}_i}^{\boldsymbol{\beta}_i} \frac{\prod_{i=1}^m (z - \mathbf{d}_i)}{w} \mathrm{d}z = 0.$$

We define the function

$$B(z, w) = e^{\int_{\mathbf{a}_0}^p \mathrm{d}j_3}. \tag{1.8}$$

Lemma 1.7. *The Chebyshev polynomial $P_{n,\infty}$ is given by*

$$P_{n,\infty}(z) = \frac{1}{2} \left(B(z, w)^n + B(z, w)^{-n} \right), \quad (z, w) \in \mathcal{R}_+. \tag{1.9}$$

Proof. By Abel's theorem, $\mathcal{E}$ is given up to a multiplicative constant uniquely by its divisor. That is,

$$\mathcal{E}(z,w) = CB(z,w)^n.$$

Since $\mathcal{E}^\tau(p) = \mathcal{E}(z,-w) = CB(z,w)^{-n}$, we obtain by (1.3), that

$$P_{n,\infty}(z) = C\frac{1}{2}\left(B(z,w)^n + B(z,w)^{-n}\right), \quad (z,w)\in\mathcal{R}_+.$$

Since $P_{n,\infty}(\mathbf{a}_0) = 1$, we see that $C = 1$. □

We would like to give an interpretation of the function B in terms of potential theory and in terms of conformal mappings. Let $\{\omega_i\}_{i=1}^m$, $\omega_i \in (0,1)$, $\omega_i \neq \omega_j$ for $i \neq j$ and $\{h_i\}_{i=1}^m$, $h_i > 0$. To this sequence we associate a *comb-domain*

$$\Pi = \{\theta = \xi + i\eta :\ -\pi < \xi < 0, \eta > 0\} \setminus \bigcup_i \{-\omega_i\pi + i\eta :\ 0 < \eta \le h_i\},$$

see survey [25], see also [39, 40]. We call a comb-domain a *n-comb* or *n-regular*, if

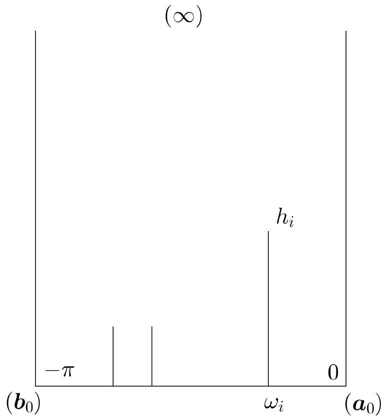


Figure 1.3: comb-domain

$n\omega_i \in \mathbb{Z}$ for all i. By $G_\Omega(z,\infty)$ we denote the potential theoretic Green's function of the domain Ω with logarithmic pole at ∞.

Lemma 1.8. *Define*

$$\theta_n(z) = -i\int_{\mathbf{a}_0}^p \mathrm{d}j_3, \quad (z,w)\in\mathcal{R}_+, \operatorname{Im} z > 0.$$

Then θ_n is a conformal mapping of the upper half-plane onto a n-comb, where the slits corresponds to the gaps $(\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j)$. Moreover,

$$\operatorname{Im}\theta_n(z) = G_{\Omega_n}(z,\infty) \tag{1.10}$$

and

$$\frac{1}{\pi}\mathrm{d}\theta_n(x) = \omega(\infty, \mathrm{d}x; \Omega_n). \tag{1.11}$$

Proof. The condition that the A-periods of $\mathrm{d}j_3$ vanish implies $\theta_n(\boldsymbol{\alpha}_i) = \theta_n(\boldsymbol{\beta}_i)$ for all $1 \le i \le m$. Hence, due to the Schwarz-Christoffel transformation formula, we see that θ_n is a conformal mapping of the upper half-plane into a comb domain. The length of the comb of π follows by the residue theorem and the fact that $\mathrm{Res}_{\infty_+} \mathrm{d}j_3 = 1$. By the reflection principle, $\theta_n(z)$ can be extended to a multi-valued function on Ω_n, but $\mathrm{Im}\,\theta_n(z)$ is a single-valued harmonic function with a logarithmic pole at infinity. That is, we have (1.10). To prove (1.11), define

$$H(z) = \frac{-1}{\sqrt{(z-\mathbf{a}_0)(z-\mathbf{b}_0)}} \prod_{i=1}^{m} \frac{z-\mathbf{d}_i}{\sqrt{(z-\boldsymbol{\alpha}_i)(z-\boldsymbol{\beta}_i)}}. \tag{1.12}$$

The proof follows the one in [56, Theorem 5.5.22]. Due to [38, Chap VII], it is a Nevanlinna class function with

$$H(z) = -\frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

at infinity. Hence, there exists an absolutely continuous probability measure ν such that

$$H(z) = \int_{E_n} \frac{\mathrm{d}\nu(x)}{x-z}$$

and

$$\mathrm{d}\nu(x) = \chi_{E_n}(x)\frac{1}{\pi}\frac{-1}{\sqrt{(x-\mathbf{a}_0)(\mathbf{b}_0-x)}} \prod_{i=1}^{m} \frac{x-\mathbf{d}_i}{\sqrt{(x-\mathbf{a}_i)(x-\mathbf{b}_i)}}\mathrm{d}x.$$

Define

$$\Phi_\nu(z) = -\int_{E_n} \log|x-z|\mathrm{d}\nu(x).$$

On $\mathbb{C}_+$,

$$\Phi_\nu'(z) = \mathrm{Re}\int_{E_n} \frac{\mathrm{d}\nu(x)}{x-z} = \mathrm{Re}\,H(z).$$

Thus, $\Phi_\nu(z)$ is constant on each connected component of E_n. Since the A-periods of $\mathrm{d}j_3$ vanish, we see that the integral of Φ_ν' over each gap vanishes. Therefore, Φ_ν is constant on E_n. That is, $\mathrm{d}\nu$ is the equilibrium measure of E_n and we get (1.11). □

Let us summarize the observations of the previous paragraph.

Theorem 1.9. *Let*

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j).$$

Then the following are equivalent:

(i) $E = E_n$,

(ii) $n\omega(\infty, E^k; \Omega) \in \mathbb{Z}$ *for all* $1 \le k \le g$,

(iii) The associated comb Π *is* n*-regular.*

In this case, the Chebyshev polynomial can be given by

$$P_{n,\infty}(z) = \cos n\theta(z).$$

Proof. We have already seen that $(i) \implies (ii), (iii)$. If (iii) holds, we can extend θ to a multi-valued function on Ω with a logarithmic pole at infinity. Let γ be a closed curve in Ω. Then, since Π is a n-comb and gaps corresponds to slits, we see that $n(\theta(z) - \theta(\gamma(z))) \in 2\pi\mathbb{Z}$. In particular, $\theta(\overline{z}) + \overline{\theta(z)} \in 2\pi\mathbb{Z}$. Hence, $\cos(n\theta(z))$ is an entire function. By considering the behaviour at infinity, we see that it is a polynomial of degree n. Counting the ± 1 points, we obtain that $\cos(n\theta(z)) = P_{n,\infty}(z)$. That is, we have (i). By (1.11), we get that $(ii) \implies (iii)$. □

Remark 1.10. (i) We call sets with the property $E = E_n$ *n-regular*. The previous theorem allows us to give a parametric description of these sets. Take any n-comb, Π_n. Due to the Riemann mapping theorem, Π_n can be conformally mapped onto $\mathbb{C}_+$, such that infinity is mapped to infinity (one gets uniqueness by e.g. the normalization that $-\pi \mapsto -1$ and $0 \mapsto 1$. All cases follow then by translation and dilation). The inverse map $\theta_n : \mathbb{C}_+ \to \Pi_n$ can be continuously extended to the boundary and $E_n := \theta_n^{-1}([-\pi, 0])$.

(ii) We would like to make a connection to Proposition 1.4. We have

$$\theta_n([\mathbf{u}_i, \mathbf{v}_i]) \subset \left[-\frac{(k+1)\pi}{n}, -\frac{k\pi}{n}\right]$$

for some $0 \le k < n$. Equality can only happen in the cases (i) and (iv), but generically (this means in this case for n large enough) equality only holds in the case (iv). Note that Lemma 1.8, implies that

$$\omega([\mathbf{u}_i, \mathbf{v}_i], \infty; \Omega_n) \le \frac{1}{n}.$$

Example 1.11. Let $E = [-2, 2]$ and let $\Pi = \{\theta = \xi + i\eta : -\pi < \xi < 0, \eta > 0\}$. Then $\zeta = e^{i\theta}$ maps Π onto $\mathbb{D} \cap \mathbb{C}_-$ and $(-\pi, 0)$ onto $\mathbb{T} \cap \mathbb{C}_-$. Hence, composing it with the Joukowski map $z = \zeta + \zeta^{-1}$, we obtain a mapping from Π onto $\mathbb{C}_+$, such that the base of the comb corresponds to the interval $[-2, 2]$. Theorem 1.9 implies

$$P_{n,\infty}(z) = \cos(n\theta), \quad z = 2\cos(\theta),$$

i.e., the well known Chebyshev polynomials for the interval $[-2, 2]$.

1.1.2 Kolmogorov's theorem

Let still $E \subset \mathbb{R}$ be compact and $x_0 \in \mathbb{R} \setminus E$. A modification of the proof of Theorem 1.3 would also give a characterization of the following closely related problem:

$$\inf\{\|P_n\|_E : P_n \in \mathcal{P}_n, P_n(x_0) = 1\}. \tag{1.13}$$

Note that this is carried out in the proof of Theorem 4.26. But if we consider non-symmetric problems, e.g., $E \subset \mathbb{R}$ but $x_0 \in \mathbb{C} \setminus \mathbb{R}$ these methods fail. A possible substitute is Kolmogorov's theorem. We demonstrate this in Section 4.2.5.

Let K be a compact subset of $\mathbb{C}$, $C(K)$ the space of continuous functions on K and V a finite dimensional subspace of $C(K)$.

Theorem 1.12. *(Kolmogorov [34]) An element $p^* \in V$ is the best approximation of $f \in C(K)$ if and only if*

$$\min_{z\in\mathcal{Z}} \operatorname{Re} \overline{(f(z)-p^*(z))}q(z) \le 0, \quad \text{for all } q \in V, \tag{1.14}$$

where $\mathcal{Z} = \{z \in K : \ |f(z)-p^*(z)| = \|f-p^*\|_K\}$.

Proof. Let us assume that (1.14) holds. Then there exists $z_0 \in \mathcal{Z}$ such that $\operatorname{Re} \overline{(f(z_0)-p^*(z_0))}q(z_0) \le 0$. Set $p = p^* + q$ for arbitrary $q \in V$. Then

$$\begin{aligned}|f(z_0)-p(z_0)|^2 &= |f(z_0)-p^*(z_0)|^2 + |q(z_0)|^2 - 2\operatorname{Re}\overline{(f(z_0)-p^*(z_0))}q(z_0)\\ &\ge |f(z_0)-p^*(z_0)|^2 = \|f-p^*\|_K.\end{aligned}$$

Hence, p approximates f not better then p^*. Conversely, assume that there exists $q \in V$ such that

$$\min_{z\in C} \operatorname{Re}\overline{(f(z)-p^*(z))}q(z) > 0.$$

Therefore, there exists $\epsilon > 0$ such that for all $z \in \mathcal{Z}$

$$\operatorname{Re}\overline{(f(z)-p^*(z))}q(z) \ge 2\epsilon.$$

By continuity, there exists an open subset $\mathcal{U}$ of K, $\mathcal{U} \supset \mathcal{Z}$, such that

$$\operatorname{Re}\overline{(f(z)-p^*(z))}q(z) > \epsilon \quad \text{for all } z \in \mathcal{U}.$$

Set $p = p_* + \lambda q$ for some $0 < \lambda < \frac{\epsilon}{\|q\|_K^2}$. Then for all $z \in \mathcal{U}$ we have

$$\begin{aligned}|f(z)-p(z)|^2 &\le \|f-p^*\|_K^2 + \lambda^2\|q\|_K^2 - 2\lambda\operatorname{Re}\overline{(f(z)-p^*(z))}q(z)\\ &< \|f-p^*\|_K^2 + \lambda^2\|q\|_K^2 - 2\lambda\epsilon\\ &< \|f-p^*\|_K^2 - \lambda\epsilon.\end{aligned}$$

Now let $F = K\setminus\mathcal{U}$, which is compact. Hence, there exist $\delta > 0$ such that $\sup_{z\in F}|f(z)-p^*(z)| < \|f-p^*\|_K - 2\delta$. Take now $\lambda < \frac{\delta}{\|q\|_K}$. Then we see that for all $z \in F$

$$|f(z)-p(z)| \le |f(z)-p^*(z)| + \lambda\|q\|_K < \|f-p^*\|_K - \delta.$$

Hence, p approximates f better then p^*. □

As application we would like to prove an alternation theorem for (1.13).

Corollary 1.13. *Let $E \subset \mathbb{R}$ be compact and $x_0 \in \mathbb{R}\setminus E$. P_n is a solution of (1.13) if and only if P_n has $n+1$ points of alternation.*

Proof. Let us assume without loss of generality that $x_0 < \inf E$. In this case (1.14) reads

$$\min_{x\in\mathcal{Z}} P_n(x)Q_{n-1}(x) \le 0 \tag{1.15}$$

for all $Q_{n-1} \in \mathcal{P}_{n-1}$. Let $x_1 < \cdots < x_k$ be a set of alternation points for P_n. Assume $k < n+1$. For $1 < j < k$, let

$$x_j^- = \min\{x \in E : x > x_{j-1}, P_n(x) = P_n(x_j)\},$$
$$x_j^+ = \max\{x \in E : x < x_{j+1}, P_n(x) = P_n(x_j)\}$$

and equivalently x_1^+ and x_k^-. We can construct a polynomial $Q \in \mathcal{P}_{n-1}$ such that $Q(x_1^+) = P_n(x_1^+)$ and for $j = 1, \ldots, k-1$, Q has a simple zero at $(x_j^+ + x_{j+1}^-)/2$. Clearly, this polynomial has the same sign as P_n at each point in $\mathcal{Z}$ and hence it violates (1.15). The other direction can be proved in the same way as in the proof of Theorem 1.3. □

1.1.3 Asymptotics of Chebyshev polynomials associated to a finite system of Jordan curves

A classical question in constructive approximation is to find asymptotics of the approximation error, $\|T_n\|_E$, and asymptotics of the extremal function, T_n. Let us first assume that E is a Jordan region with smooth boundary (i.e., E is bounded by a smooth image of the unit circle). Hence, $\Omega = \overline{\mathbb{C}} \setminus E$ is simply connected. Let $B_\Omega(z,\infty)$ denote the unique Riemann map that maps Ω onto the unit disc $\mathbb{D}$ such that $B_\Omega(\infty,\infty) = 0$ fixed by the normalization that the derivative at infinity is positive, i.e., $(zB_\Omega(z,\infty))(\infty) > 0$. This value is called the capacity of the domain

$$\operatorname{Cap}(E) := |zB_\Omega(z,\infty)|(\infty).$$

In this situation, the above mentioned questions were answered by Faber [26], see also Bernstein [14].

Theorem 1.14. *Let E be a Jordan region with smooth boundary. Then*

$$\|T_n\|_E \sim \operatorname{Cap}(E)^n \tag{1.16}$$

and

$$\lim_{n\to\infty} \frac{T_n(z)B_\Omega(z,\infty)^n}{\operatorname{Cap}(E)^n} = 1, \tag{1.17}$$

uniformly on compact subsets of Ω.

Note that (1.17) is an easy consequence of (1.16), the maximum principle and Montel's theorem. Following ideas of Akhiezer [4, 5], Widom, in his seminal work on asymptotics of extremal polynomials in the complex plane [67], found the right generalization for (1.17) in multiply connected domains.

Let $E = \bigcup_{k=0}^g E_k$ be a finite union of smoothly bounded disjoint Jordan regions and smooth Jordan arcs. Moreover, let $G_\Omega(z,z_0)$ denote the Green's function of the point z_0 and the domain Ω. Writing $i * G_\Omega(z,z_0)$ for the harmonic conjugate of $G_\Omega(z,z_0)$, we define the complex Green's function of the domain by

$$B_\Omega(z,z_0) = e^{-(G_\Omega(z,z_0)+i*G_\Omega(z,z_0))}.$$

Remark 1.15. We would like to point out a connection to the previous section. If E is a finite system of intervals and $B(z,w)$ the function defined by (1.8) on the associated Riemann surface, then, by Lemma 1.8,

$$B_\Omega(z,\infty) = B(z,w), \quad (z,w) \in \mathcal{R}_+.$$

A lower bound for $\|T_n\|_E$ is already due to Szegő [60]. Due to the maximum principle and properties of the Green's function we see that

$$\|T_n\|_E = \|T_n B_\Omega(\cdot,\infty)^n\|_E \geq |T_n(z)B_\Omega(z,\infty)^n|(\infty) = \mathrm{Cap}(E)^n. \tag{1.18}$$

Note that B_Ω is not a single-valued function in this case. In fact, let us choose generators $\tilde{\gamma}_1,\dots,\tilde{\gamma}_g$ of the fundamental group $\pi(\Omega;\infty)$, which encircles only the set E_k once. Then

$$B_\Omega(\tilde{\gamma}_k(z),\infty) = e^{2\pi i\omega(\infty,E_k;\Omega)}B_\Omega(z,\infty),$$

see e.g. [3, Chapter 6]. Let $\pi(\Omega)^*$ denote the group of additive characters of $\pi(\Omega;\infty)$. Given a character $\alpha \in \pi(\Omega)^*$, we define the Hardy space of character automorphic functions by

$$H^\infty_\Omega(\alpha) := \{F:\Omega\to\mathbb{C}:\ F \text{ is analytic and bounded in } \Omega,\ F\circ\gamma = e^{2\pi i\alpha(\gamma)}F\}.$$

If one works with multi-valued functions, it is convenient to use the universal cover of Ω. Due to the uniformization theorem [47], there exists a Fuchsian group Γ such that $\Omega \simeq \mathbb{D}/\Gamma$. This means there exists a function $\mathfrak{z}:\mathbb{D}\to\Omega$, which is onto, such that $\mathfrak{z}(\zeta_1) = \mathfrak{z}(\zeta_2)$ implies that $\zeta_1 = \gamma(\zeta_2)$ for some $\gamma\in\Gamma$. This map is unique under the normalization condition $\mathfrak{z}(0)=\infty$ and $\mathfrak{z}'(0)>0$. Note that the transformation $\gamma\in\Gamma$ corresponds to an element in $\pi(\Omega;\infty)$, in fact, $\Gamma\simeq\pi(\Omega;\infty)$. We will describe this relation in more detail in Section 4.2.1. By

$$\Gamma^* = \{\alpha \,|\, \alpha\colon \Gamma\to\mathbb{R}/\mathbb{Z} \text{ such that } \alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)+\alpha(\gamma_2)\}$$

we denote the group of characters of Γ. Clearly $\Gamma^*\simeq\pi(\Omega)^*$ and since Γ acts freely, $\Gamma^*\cong\mathbb{R}^g/\mathbb{Z}^g$. For $\alpha\in\Gamma^*$, we define the Hardy space of character automorphic functions by

$$H^\infty(\alpha) = \left\{f\in H^\infty(\mathbb{D})\colon f\circ\gamma = e^{2\pi i\alpha(\gamma)}f,\ \gamma\in\Gamma\right\}, \tag{1.19}$$

with the standard sup-norm, $\|\cdot\|_\infty$, on $\mathbb{D}$. The operator $F\mapsto f := F\circ\mathfrak{z}$ shows that $H^\infty_\Omega(\alpha)\simeq H^\infty(\alpha)$.

Example 1.16. Fix $z_0\in\Omega$ and consider the associated orbit $\mathrm{orb}(\zeta_0) = \mathfrak{z}^{-1}(z_0) = \{\gamma(\zeta_0)\colon \gamma\in\Gamma\}$. The Blaschke product $b_{z_0}(\zeta)$ with zeros at $\mathfrak{z}^{-1}(z_0)$ is called the Green's function of the group Γ; cf. [51]. Let $\zeta_0\in\mathfrak{z}^{-1}(z_0)$. Then

$$b_{z_0}(\zeta) = \prod_{\gamma\in\Gamma}\frac{|\gamma(\zeta_0)|}{\gamma(\zeta_0)}\frac{\gamma(\zeta_0)-\zeta}{1-\overline{\gamma(\zeta_0)}\zeta},$$

with the appropriate modification if $z_0=\infty$ and $\gamma(\zeta_0)=0$. It is normalized so that $b_{z_0}(0)>0$ if $z_0\neq\infty$ and $(\mathfrak{z}b_\infty)(0)>0$. Then $B_\Omega(\mathfrak{z}(\zeta),z_0) = b_{z_0}(\zeta)$. We denote the corresponding character by μ_{z_0}. If $z_0=\infty$ we use the abbreviations $b(\zeta)=b_\infty(\zeta)$ and $\mu=\mu_\infty$.

The asymptotics of T_n will be described by means of a corresponding character automorphic H^∞ problem.

Problem 1. *Let $\alpha \in \Gamma^*$. Find:*

$$\inf\{\|w\|_\infty : \ w \in H^\infty(\alpha), \ w(0) = 1\}.$$

In a slightly different language, that is, without using coverings, already Widom constructed a solution of this problem.

Theorem 1.17 ([67]). *Let E be a finite union of disjoint Jordan regions and arcs. Then for each $\alpha \in \Gamma^*$ there is a unique solution, w_α, for Problem 1.*

Let $\mu(\gamma_k) = \omega(\infty, E_k; \Omega)$ be the character of $B_\Omega(z, \infty)$. Then

$$\frac{T_n(z)B_\Omega(z,\infty)^n}{\mathrm{Cap}(E)^n} \in H^\infty_\Omega(n\mu), \quad \frac{T_n(z)B_\Omega(z,\infty)^n}{\mathrm{Cap}(E)^n}(\infty) = 1.$$

Hence, again we easily get a lower bound for $\|T_n\|_E$, which improves (1.18), i.e., by definition we see that

$$\|T_n\|_E \geq \mathrm{Cap}(E)^n \|w_{n\mu}\|_\infty. \tag{1.20}$$

Under the restriction that E has no arc components, Widom was able to show that (1.20) indeed describes the right rate of convergence and accomplished to prove an analog of Theorem 1.14.

Theorem 1.18. *Let E be a finite union of Jordan regions. Then*

$$\|T_n\|_E \sim \mathrm{Cap}(E)^n \|w_{n\mu}\|_\infty$$

and

$$\lim_{n\to\infty} \left| \frac{T_n(\mathfrak{z}(\zeta))b(\zeta)^n}{\mathrm{Cap}(E)^n} - w_{n\mu}(\zeta) \right| = 0,$$

uniformly on compact subsets of $\mathbb{D}$.

An analogous result for system of arcs and regions is still not known. But the situation is much better for real sets E.

1.1.4 Asymptotics of Chebyshev polynomials associated to a finite system of intervals

In [67], Widom was only working in the finitely connected setting. Hence, $E \subset \mathbb{R}$ means that E is a finite union of intervals. Let us again write it in the form (1.1). The statement of Theorem 1.17 can be specified in this case. See e.g. Proposition 4.16 or [66, Lemma 6.4], where this was proved in a much higher generality. See also Proposition 4.23.

Theorem 1.19. *Let $\alpha \in \Gamma^*$. There exists a collection of points $\mathfrak{X} = \{\mathbf{x}_i\}_{i=1}^g$, $\mathbf{x}_i \in [\mathbf{a}_i, \mathbf{b}_i]$ such that the solution of Problem 1 is given by*

$$w_\alpha = \prod_{i=1}^g \frac{b_{\mathbf{x}_i}}{b_{\mathbf{x}_i}(0)},$$

where we set $b_{\mathbf{x}_i} = 1$ if $\mathbf{x}_i = \mathbf{a}_i$ or $\mathbf{x}_i = \mathbf{b}_i$. Moreover, with the identification of the endpoints $\mathfrak{X}$ is unique.

In fact, if $\times_{i=1}^g [\mathbf{a}_i, \mathbf{b}_i]$ is equipped with the product topology of circles, then the map $\alpha \mapsto \mathfrak{X}$ is a homeomorphism.

Widom already proved that:

Theorem 1.20. *Let E be a finite system of intervals. Then*

$$\|T_n\|_E \sim 2\,\mathrm{Cap}(E)^n \|w_{n\mu}\|_\infty.$$

Recently, Christiansen, Simon and Zinchenko also proved the corresponding asymptotics of T_n [16].

Theorem 1.21. *Let E be a finite union of intervals. Then*

$$\|T_n\|_E \sim 2\,\mathrm{Cap}(E)^n \|w_{n\mu}\|_\infty$$

and

$$\lim_{n\to\infty} \left| \frac{T_n(\mathfrak{z}(\zeta))b(\zeta)^n}{\mathrm{Cap}(E)^n} - w_{n\mu}(\zeta) \right| = 0,$$

uniformly on compact subsets of $\mathbb{D}$.

Remark 1.22.

- Meanwhile, jointly with Yuditskii, this theorem is generalized to regular and compact $E \subset \mathbb{R}$, which are of Widom type such that the direct Cauchy theorem holds. All finite union of intervals are of this type, but generically in this case Ω is infinitely connected. See Remark 1.34 for the definition. Since this thesis only deals with finitely connected domains, we will not discuss this result here in detail.

1.2 Orthogonal Polynomials

1.2.1 Asymptotics of Orthogonal Polynomials

In the same paper [67] Widom also considered asymptotics of orthogonal polynomials. Let E be a finite system of smooth Jordan curves and arcs and Ω be the exterior of this system. Let σ be a finite measure on E. The monic polynomial of degree at most n minimizing the value

$$\int_E |P_n|^2 \mathrm{d}\sigma,$$

is the nth monic orthogonal polynomial with respect to the measure $\mathrm{d}\sigma$. Let

$$\|P_n\|_{2,\sigma,P}^2 = \int_E |P_n|^2 \mathrm{d}\sigma,$$

where along arcs the integral is only taken once. In the following, let us assume that $d\sigma(z) = \rho(z)|dz|$, where $\rho > 0$ is assumed to be continuously differentiable. In order to study asymptotics of P_n, Widom again first solved a corresponding character automorphic problem for analytic functions. But naturally, H^∞ should be substituted by H^2. Similar as $H^\infty(\alpha)$ we define

$$H^2_\Omega(\alpha) := \{F : \Omega \to \mathbb{C} :\ |F|^2 \text{ has an harmonic majorant },\ F \circ \gamma = e^{2\pi i\alpha(\gamma)}F\}$$

and the corresponding Hardy space on the universal cover by

$$H^2(\alpha) = \Big\{ f \in H^2(\mathbb{D})\colon f \circ \gamma = e^{2\pi i\alpha(\gamma)} f,\ \gamma \in \Gamma \Big\}, \tag{1.21}$$

with the standard norm on $H^2(\mathbb{D})$

$$\|f\|_2^2 = \int_{-\pi}^{\pi} |f(\zeta)|^2 \frac{d\zeta}{2\pi}.$$

Again we have $H^2_\Omega(\alpha) \simeq H^2(\alpha)$ by the map $F \mapsto f := F \circ \mathfrak{z}$. The corresponding norm in $H^2_\Omega(\alpha)$ is the value of the least harmonic majorant at infinity. For Hardy spaces of single-valued functions in multiply connected domains we refer to [55]. The corresponding $H^2_\Omega(\alpha)$ problem is given by:

Problem 2.

$$\inf\{\|F\|_{2,\sigma} :\ f \in H^2_\Omega(\alpha),\ F(\infty) = 1\},$$

where

$$\|F\|^2_{2,\sigma} = \int_E |F|^2 d\sigma.$$

Note that clearly here also on arc components, E_k, we have to compute

$$\oint_{E_k} |F|^2 d\sigma,$$

because generally speaking the boundary values of F on the two sides of the arc do not coincide. Widom proved that this problem has a unique solution, which we will denote by F_α and the corresponding function on the universal cover will be denoted by f_α. In this case Widom gave a complete description (for curves and arcs!) of the asymptotics.

Theorem 1.23. *Let E and σ be as above. Then*

$$\|P_n\|_{2,\sigma,P} \sim \mathrm{Cap}(E)^n \|F_{n\mu}\|_{2,\sigma},$$

$$\lim_{n\to\infty} \oint_E \left| \frac{T_n(z)}{\mathrm{Cap}(E)^n} - \frac{F_{n\mu}(z)}{B_\Omega(z,\infty)^n} \right|^2 d\sigma(z) = 0$$

and

$$\lim_{n\to\infty} \left| \frac{P_n(\mathfrak{z}(\zeta)) b(\zeta)^n}{\mathrm{Cap}(E)^n} - f_{n\mu}(\zeta) \right| = 0,$$

uniformly on compact subsets of $\mathbb{D}$.

1.2.2 Orthogonal polynomials for compactly supported real measures

Let $\mathrm{d}\sigma_+$ be a real, scalar, compactly supported probability measure and $\{P_n(x)\}_{n\geq 0}$ the corresponding orthonormal polynomials, which we obtain by orthonormalizing the monomials

$$1,\, x,\, x^2,\, \ldots .$$

To indicate the dependence on the measure, we may also write $P_n(x, \mathrm{d}\sigma_+)$. It is easy to see that they obey

$$\begin{aligned} xP_n(x) &= a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \qquad a_n > 0, b_n \in \mathbb{R}, \\ xP_0(x) &= a_1P_1(x) + b_0P_0(x), \end{aligned}$$

that is, the multiplication by the independent variable in the basis $\{P_n(x)\}_{n\geq 0}$ has the matrix

$$J_+ = \begin{bmatrix} b_0 & a_1 & 0 & \\ a_1 & b_1 & a_2 & \\ 0 & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

where $|a_n|, |b_n| \leq C$ for C such that $\mathrm{d}\sigma_+$ has support $[-C, C]$. Matrices of this sort are called one-sided (bounded) Jacobi matrices. In general, we call an operator one-sided if it is an operator on $\ell^2_+ = \ell^2(\mathbb{Z}_{\geq 0})$ and correspondingly two-sided if it acts on $\ell^2 = \ell^2(\mathbb{Z})$. Associated to $\mathrm{d}\sigma_+$ we also define the orthogonal polynomials of the second kind, by

$$Q_n(z) = \int \frac{P_n(x) - P_n(z)}{x - z}\, \mathrm{d}\sigma_+(x), \quad n \geq 0.$$

Let $P_n(z, \mathrm{d}\sigma_+^{(1)})$ denote the polynomials associated to the shifted Jacobi matrix $J_+^{(1)} = S_+^* J_+ S_+$, where $S_+ e_n = e_{n+1}$ denotes the right shift on ℓ^2_+. Then by showing that Q_n also solves a three-term recursion relation, one obtains

$$Q_n(z, \mathrm{d}\sigma_+) = \frac{1}{a_1} P_{n-1}(z, \mathrm{d}\sigma_+^{(1)}).$$

Moreover, we define the so-called elementary Blaschke-Potapov factor by

$$\mathfrak{a}^J(z, a, b) := \begin{bmatrix} 0 & a \\ -\frac{1}{a} & \frac{z-b}{a} \end{bmatrix},$$

see [52]. Hence, we see that

$$\begin{bmatrix} a_{n+1}P_n(z) \\ P_{n+1}(z) \end{bmatrix} = \mathfrak{a}^J(z, a_{n+1}, b_n) \begin{bmatrix} a_nP_{n-1}(z) \\ P_n(z) \end{bmatrix}.$$

Defining the *transfer matrix*

$$\mathfrak{A}_n^J(z) = \begin{bmatrix} \mathfrak{A}_n^J(z)_{1,1} & \mathfrak{A}_n^J(z)_{1,2} \\ \mathfrak{A}_n^J(z)_{2,1} & \mathfrak{A}_n^J(z)_{2,2} \end{bmatrix} = \mathfrak{a}^J(z, a_n, b_{n-1}). \ldots .\mathfrak{a}^J(z, a_1, b_0),$$

this implies

$$\begin{bmatrix} a_nP_{n-1}(z) \\ P_n(z) \end{bmatrix} = \mathfrak{A}_n^J(z) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

1.2.3 Direct spectral theory of periodic Jacobi matrices

If the coefficient sequences are periodic, it is convenient to extend them periodically on $\mathbb{Z}$ and consider the associated two-sided Jacobi matrix, J, as an operator on ℓ^2, i.e.,

$$Je_n = a_{n+1}e_{n+1} + b_n e_n + a_n e_{n-1},$$

where $\{e_n\}_{n\in\mathbb{Z}}$ denotes the standard basis of ℓ^2. Moreover, let $\ell^2_- = \ell^2 \ominus \ell^2_+$ with the classical embedding of ℓ^2_+ into ℓ^2. Since $a_k > 0$, the subspace $\{e_{-1}, e_0\}$ is cyclic for J. Let $J_\pm = P_\pm J P_\pm^*$, where $P_\pm$ denotes the orthogonal projector onto $\ell^2_\pm$. We define the resolvent matrix functions related to the two-sided Jacobi matrix

$$R^J(z) = \begin{bmatrix} R^J_{-1,-1} & R^J_{-1,0} \\ R^J_{0,-1} & R^J_{0,0} \end{bmatrix}(z), \quad \text{where } R^J_{n,m}(z) = \langle (J-z)^{-1}e_m, e_n\rangle$$

and resolvent functions related to the one-sided matrices

$$r^J_-(z) = \langle (J_- - z)^{-1}e_{-1}, e_{-1}\rangle, \quad r^J_+(z) = \langle (J_+ - z)^{-1}e_0, e_0\rangle.$$

Since,

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + a_0(\langle \cdot, e_0\rangle e_{-1} + \langle \cdot, e_{-1}\rangle e_0), \tag{1.22}$$

one can show that

$$R^J(z) = \begin{bmatrix} r^J_-(z)^{-1} & a_0 \\ a_0 & r^J_+(z)^{-1} \end{bmatrix}^{-1}.$$

From this we obtain

$$-\frac{1}{R^J_{-1,-1}(z)} = a_0^2 r^J_+(z) - \frac{1}{r^J_-(z)}, \quad -\frac{1}{R^J_{0,0}(z)} = a_0^2 r^J_-(z) - \frac{1}{r^J_+(z)}, \tag{1.23}$$

$$R^J_{0,0}(z) = R^J_{-1,-1}(z)\frac{r^J_+(z)}{r^J_-(z)}. \tag{1.24}$$

Note that we did not use periodicity so far. Let $r^{J^{(1)}}_\pm$ be the resolvent functions related to the shifted Jacobi matrix $J^{(1)} = S^{-1}JS$, where $Se_n = e_{n+1}$ denotes the right shift on ℓ^2. Then, using the Sherman–Morrison–Woodbury formula (cf. [29, Section 2.1.3]) we see that

$$r^J_+(z) = \frac{-1}{z - b_0 + a_1^2 r^{J^{(1)}}_+(z)}, \quad r^{J^{(1)}}_-(z) = \frac{-1}{z - b_0 + a_0^2 r^J_-(z)}. \tag{1.25}$$

Iterating the first identity leads to

$$r^J_+(z) = \frac{\mathfrak{A}^J_n(z)_{1,1} r^{J(n)}_+(z) + \mathfrak{A}^J_n(z)_{2,1}}{\mathfrak{A}^J_n(z)_{1,2} r^{J(n)}_+(z) + \mathfrak{A}^J_n(z)_{2,2}} \tag{1.26}$$

In the previous paragraph we presented all ingredients for the proof of the following theorem:

Theorem 1.24. *Let J be a two-sided p-periodic Jacobi matrix. Define the discriminant $\Delta_p^J(z) = \operatorname{tr}\mathfrak{A}_p^J(z)$. Then the spectrum of J is purely absolutely continuous of multiplicity 2 and it is given by*

$$\sigma(J) = \sigma_{ac}(J) = \{z \in \mathbb{C} : \Delta_p^J(z) \in [-2, 2]\}. \tag{1.27}$$

Moreover, the resolvent function is given by

$$r_+^J(z) = -\frac{P_p(z) + a_0 Q_{p-1}(z) \pm \sqrt{\Delta_p^J(z)^2 - 4}}{2a_0 P_{p-1}(z)}$$

and r_+^J and r_-^J obey

$$a_0^2 r_+^J(x + i0) = \frac{1}{r_-^J(x - i0)}, \qquad \textit{for all} \;\; x \in \sigma(J). \tag{1.28}$$

Remark 1.25. We will see (Proposition 1.32) in the next section that if a finite system of intervals, E, is the spectrum of a periodic Jacobi matrix, J, then $\Delta_p^J = 2P_{p,\infty}$, where $P_{p,\infty}$ is the associated normalized Chebyshev polynomial. Hence, Δ_p^J has 1) only real critical points, 2) $|\Delta_p^J(c)| \geq 2$ for all critical points c.

Example 1.26. Let J_+^0 be the one-sided free discrete Schrödinger operator defined by

$$J_+^0 = \begin{bmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ 0 & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \end{bmatrix}.$$

It's discriminant is given by

$$\Delta_1^J(z) = \operatorname{tr}\begin{bmatrix} 0 & 1 \\ -1 & z \end{bmatrix} = z$$

Therefore,

$$r_+^J(z) = -\frac{z - \sqrt{z^2 - 4}}{2} = \int_{-2}^{2} \frac{\mathrm{d}\sigma_+^0}{x - z},$$

where

$$\mathrm{d}\sigma_+^0 = \sqrt{4 - x^2}\frac{\mathrm{d}x}{2\pi}.$$

The measure σ_+^0 is called Wigner's semicircle distribution and plays an important role in the theory of random matrices [69]. Hence, J_+^0 has purely absolutely continuous spectrum, which is the interval $[-2, 2]$. It follows from the recursion relation that the orthogonal polynomials are given by

$$P_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Relation (1.28) is characteristic in the following sense:

Definition 1.27. Let $A \subset \mathbb{R}$ be of positive Lebesgue measure. A Jacobi matrix J is called *reflectionless* on A, if

$$a_0^2 r_+^J(x+i0) = \frac{1}{r_-^J(x-i0)}, \qquad \text{for a.e. } \; x \in A. \tag{1.29}$$

Let J be a p-periodic Jacobi matrix and

$$\sigma(J) = E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j),$$

where $g \leq p-1$. Then the isospectral set

$$J(E) = \{J \colon \sigma(J) = E \text{ and } J \text{ is reflectionless on } E\} \tag{1.30}$$

consists of all periodic Jacobi matrices with the given spectrum. The following corollary is apparent:

Corollary 1.28. *Let J be a p-periodic Jacobi matrix, $\Delta_p^J(z)$ the associated discriminant and $E = \sigma(J)$. Let $\{a_j, b_j\}_{j=1}^p$ be a set of Jacobi parameters, $\tilde{J} = \tilde{J}(\{a_j, b_j\}_{j=1}^p)$ the induced two-sided periodic Jacobi matrix and $\tilde{\Delta}(z) = \operatorname{tr} \mathfrak{A}_p^{\tilde{J}}(z)$ the corresponding discriminant. Then $\tilde{J} \in J(E)$ if and only if $\Delta_p^J = \tilde{\Delta}$.*

Some heuristics: Generically (i.e., if $|\Delta_p^J(c)| > 2$ for all critical points c) the spectrum, E, is a system of p intervals. Hence, for the number of gaps, g, we have $g = p-1$. Let us count the number of free parameters. We have $2p$ parameters $\{a_j, b_j\}_{j=1}^p$, but the condition $\Delta_p^J = \tilde{\Delta}$ gives $p+1$ restrictions on it. Thus, in fact we have $p-1 = g$ free parameters. We will make this considerations explicit in the next chapter. That is, we will show that $J(E)$ is always homeomorphic to a g-dimensional real torus, where g is the number of gaps.

1.2.4 The isospectral torus and the Abel map

Let

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j)$$

be an arbitrary finite system of intervals. Note that in general E is not necessarily the spectrum of a periodic Jacobi matrix. We will see that this is only the case if there exists a polynomial T, with the properties 1) and 2) such that $E = T^{-1}([-2, 2])$. But still we can define the isospectral set, $J(E)$, by (1.30). It is well known that all elements of the isospectral torus have purely absolutely continuous spectrum of multiplicity 2.

Remling [54] proved recently that roughly speaking, up to a small perturbation, every Jacobi matrix is reflectionless on the absolutely continuous part of its spectrum.

Theorem 1.29. *Let $A \subset \mathbb{R}$ of positive Lebesgue measure be the support of the absolutely continuous spectrum of a one-sided Jacobi matrix J_+ with Jacobi parameters $\{a_n, b_n\}_{n=0}^{\infty}$. Let J^{∞} be an arbitrary right-limit of J_+, i.e.,*

$$a_n^{(\infty)} = \lim_{j\to\infty} a_{n+m_j}, \quad b_n^{(\infty)} = \lim_{j\to\infty} b_{n+m_j},$$

for all $n \in \mathbb{Z}$. Then $J^{(\infty)}$ is reflectionless on A.

Remark 1.30. (i) Note that the family $\{(S_+^*)^n J_+ S_+^n\}_n$ is compact in the pointwise topology. Thus, existence of right-limits is guaranteed.

(ii) Given an ergodic family of Jacobi matrices it was shown by Kotani-Pastur-Ishni [48] that almost all elements of this family are reflectionless on the absolutely continuous part of the spectrum. The result of Remling removes the probabilistic statement and shows that all almost periodic Jacobi matrices are reflectionless on the absolutely continuous part of the spectrum. Thus, an equivalent definition of the isospectral torus is

$$J(E) = \{J : \ \sigma(J) = \sigma_{ac}(J) = E, \ J \text{ is almost periodic}\}.$$

The map $J(E) \longleftrightarrow D(E)$

The reflectionless property allows us to associate to each Jacobi matrix $J \in J(E)$ a rational function on the corresponding Riemann surface. As in the previous section, we define

$$R_E(z) = (z - \mathbf{a}_0)(z - \mathbf{b}_0) \prod_{i=1}^{g} (z - \mathbf{a}_i)(z - \mathbf{b}_i)$$

and the induced Riemann surface $\mathcal{R}$ as in (1.5). Due to the reflectionless property, we can define a rational function, $r^J(z, w)$, by

$$r^J(z,w) = \begin{cases} a_0^2 r_+^J(z) & \text{if } (z,w) \in \mathcal{R}_+, \\ r_-^J(z)^{-1} & \text{if } (z,w) \in \mathcal{R}_-. \end{cases}$$

Indeed, from the reflectionless property it follows that

$$\operatorname{Re} R_{-1,-1}^J(x + i0) = 0 \quad \text{for almost every } x \in E,$$

which allows us, by writing the integral representation for $\log R_{-1,-1}^J$, to find that

$$R_{-1,-1}^J(z) = -\frac{1}{\sqrt{(z - \mathbf{a}_0)(z - \mathbf{b}_0)}} \prod_{i=1}^{g} \frac{z - \mathbf{x}_i}{\sqrt{(z - \mathbf{a}_i)(z - \mathbf{b}_i)}},$$

for some $\mathbf{x}_i \in [\mathbf{a}_i, \mathbf{b}_i]$, see e.g. [17, 35]. This representation allows us to find the divisor of the rational function r^J. Due to (1.23), $\mathbf{x}_i$ is a pole of at least one of the functions r_+^J or $(r_-^J)^{-1}$. By (1.24), we see that it can only be a pole of one of them, for otherwise it would be a pole of $R_{0,0}^J$. We write $(\mathbf{x}_i, 1)$ if $\mathbf{x}_i$ is a pole of

r_+, and $(\mathbf{x}_i, -1)$ if $\mathbf{x}_i$ is a pole of $(r_-^J)^{-1}$. Moreover, r^J has a zero at ∞_+ and a pole at ∞_-. Let us introduce the special class of divisors

$$D(E) = \{\{(\mathbf{x}_i, \epsilon_i)\} : \mathbf{x}_i \in [\mathbf{a}_i, \mathbf{b}_i], \epsilon_i \in \{-1, 1\}\},$$

with the identification $(\mathbf{a}_i, -1) = (\mathbf{a}_i, 1)$ and $(\mathbf{b}_i, -1) = (\mathbf{b}_i, 1)$ and equipped with the product topology of circles. To the function r^J we associate the divisor $D^{(0)} \in D(E)$ as described above. Using (1.25), we conclude that the poles of $r^{J^{(1)}}$ are zeros of r^J. Denoting the divisor of poles of $r^{J^{(1)}}$ by $D^{(1)} \in D(E)$ we see that

$$(r^J) = D^{(1)} + \infty_+ - D^{(0)} - \infty_-,$$

where (f) denotes the divisor of zeros and poles of a rational function on $\mathcal{R}$. The knowledge of the divisor and the behaviour of the resolvent functions at infinity allows to reconstruct them by means of Abelian integrals of the third kind like in (1.8). In fact, already the knowledge of $D^{(0)}$ is enough to uniquely reconstruct J. The following theorem is well known:

Theorem 1.31. *There is a one-to-one correspondence between $D(E)$ and $J(E)$. In particular, given a divisor $D \in D(E)$ the resolvent functions of the corresponding $J \in J(E)$ are given by*

$$a_0^2 r_+^J(z) = \frac{1}{2}\left(\sqrt{(z-\mathbf{a}_0)(z-\mathbf{b}_0)}\prod_{i=1}^{g}\frac{\sqrt{(z-\mathbf{a}_i)(z-\mathbf{b}_i)}}{z-\mathbf{x}_i} - (z-\alpha) + \sum_{i=1}^{g}\frac{\sigma_i\epsilon_i}{z-\mathbf{x}_i}\right),$$
$$-\frac{1}{r_-^J(z)} = \frac{1}{2}\left(\sqrt{(z-\mathbf{a}_0)(z-\mathbf{b}_0)}\prod_{i=1}^{g}\frac{\sqrt{(z-\mathbf{a}_i)(z-\mathbf{b}_i)}}{z-\mathbf{x}_i} + (z-\alpha) - \sum_{i=1}^{g}\frac{\sigma_i\epsilon_i}{z-\mathbf{x}_i}\right),$$

where

$$\sigma_k = \frac{\prod_{i=0}^{g}\sqrt{(\mathbf{x}_k-\mathbf{a}_i)(\mathbf{x}_k-\mathbf{b}_i)}}{\prod_{i\neq k}(\mathbf{x}_k-\mathbf{x}_i)}, \quad \text{and} \quad \alpha = \frac{\mathbf{a}_0+\mathbf{b}_0}{2} + \sum_{i=1}^{g}\frac{\mathbf{a}_i+\mathbf{b}_i+2\mathbf{x}_i}{2}.$$

Vice versa, a reflectionless Jacobi matrix defines the divisor, D, by the above construction.

For trace formulas for the coefficient sequences in terms of the divisor $D^{(n)}$ see e.g. [17] or [61, Section 8.3.]. Theorem 1.31 finally provides a link of the theory of real Chebyshev polynomials and the theory of reflectionless Jacobi matrices:

Theorem 1.32. *E is the spectrum of n-periodic Jacobi matrices if and only if*

$$n(\infty_+ - \infty_-),$$

is the divisor of a rational function on $\mathcal{R}$. That is, if and only if E coincides with its n-extension.

The map $\Gamma^* \longrightarrow J(E)$

In section (1.2.2) we have already seen that there is a one-to-one correspondence between $J(E)$ and $D(E)$. By virtue of the Abel map, they are also homeomorphic to $\mathbb{R}^g/\mathbb{Z}^g$ or what is equivalent, to Γ^*. Recall that $\{\mathrm{d}j_1^l\}$ denotes the canonical basis of Abelian differentials of the first kind. By the real version of Abel's theorem, the Abel map $\mathcal{A} : D(E) \to \mathbb{R}^g/\mathbb{Z}^g$, defined by

$$\mathcal{A}(\{(\mathbf{x}_i, \epsilon_i)\}) = \left\{ \sum_{i=1}^{g} \int_{(\mathbf{x}_i,\epsilon_i)}^{\mathbf{b}_i} \mathrm{d}j_1^l \right\}_{l=1}^{g}.$$

is a bijection. By means of Lemma 1.5, we see that alternatively we can write it as

$$\mathcal{A}(\{(\mathbf{x}_i, \epsilon_i)\}) = \left\{ \sum_{i=1}^{g} \frac{\epsilon_i}{2} \int_{\mathbf{x}_i}^{\mathbf{b}_i} \omega(\mathrm{d}t, E^l; \Omega) \right\}_{l=1}^{g}. \tag{1.31}$$

In order to deal with more complicated sets, such as the already mentioned Parreau-Widom sets, Sodin and Yuditskii [59] introduced a mapping $\pi : J(E) \to \Gamma^*$, which they called *generalized Abel map*. We are going to derive this map in the sequel.

First, we will develop a functional model for Jacobi matrices, where the elements of $J(E)$ will be parametrized by the Hardy spaces $H^2(\alpha)$. Recall the definition in Section 1.2.1. Since $H^2(\alpha) \subset H^2(\mathbb{D})$ it is clearly a reproducing kernel Hilbert space and we denote the kernels by $k^\alpha(\zeta, \zeta_0) = k^\alpha_{\zeta_0}(\zeta)$. That is, they obey

$$f(\zeta_0) = \langle f, k^\alpha_{\zeta_0} \rangle, \quad \forall f \in H^2(\alpha).$$

If $\zeta_0 = 0$, we use the abbreviations $k^\alpha(\zeta) = k^\alpha_0(\zeta)$. Recall that $\mathfrak{z}(0) = \infty$ and that we also agreed to write $b = b_\infty$ and $\mu = \mu_\infty$. Another important function will be the inner part of b', denoted by $\mathcal{W}$. We will refer to $\mathcal{W}$ as the *Widom function*. We have

$$\mathcal{W}(\zeta) = \prod_{i=1}^{g} b_{\mathbf{d}_i}(\zeta),$$

where $\mathbf{d}_i$ are the critical points of $G_\Omega(z, \infty)$. Let $\nu = \mu_{\mathbf{d}_1} + \cdots + \mu_{\mathbf{d}_g}$ denote its character.

Lemma 1.33. *For all $f \in H^1(\nu)$ we have*

$$\int_{-\pi}^{\pi} \frac{f(\zeta)}{\mathcal{W}(\zeta)} \frac{\mathrm{d}\zeta}{2\pi} = \langle f, \mathcal{W} \rangle = \left(\frac{f}{\mathcal{W}} \right)(0). \tag{1.32}$$

Proof. Since the Lebesgue measure of $\mathbb{T}$ corresponds to the harmonic measure on E (see also discussion in Section 4.2.2), we obtain by the Cauchy theorem for $H^1_\Omega(\mathrm{id})$ functions [55, Theorem 3.2] that

$$\int_{-\pi}^{\pi} \frac{f(\zeta)}{\mathcal{W}(\zeta)} \frac{\mathrm{d}\zeta}{2\pi} = \oint_E \frac{F(x)}{\prod B_\Omega(x, \mathbf{d}_i)} \frac{-1}{\sqrt{(x-\mathbf{a}_0)(\mathbf{b}_0 - x)}} \prod_{i=1}^{m} \frac{x - \mathbf{d}_i}{\sqrt{(x-\mathbf{a}_j)(x-\mathbf{b}_j)}} \frac{\mathrm{d}x}{2\pi}$$

$$=2\pi i \operatorname{Res}_\infty \left(\frac{F(x)}{\prod B_\Omega(x, \mathbf{d}_i)}(x) \frac{-1}{\sqrt{(x-\mathbf{a}_0)(\mathbf{b}_0 - x)}} \prod_{i=1}^m \frac{x - \mathbf{d}_i}{\sqrt{(x-\mathbf{a}_j)(x-\mathbf{b}_j)}} \frac{\mathrm{d}x}{2\pi} \right)$$
$$= \left(\frac{F}{\prod B_{\mathbf{d}_i}} \right)(\infty) = \left(\frac{f}{\mathcal{W}} \right)(0).$$

□

Remark 1.34. Let $E \subset \mathbb{R}$ be a compact, regular set of positive Lebesgue measure. Hence, we can write it as

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{i=1}^{\infty} (\mathbf{a}_i, \mathbf{b}_i).$$

The Green's function $G_\Omega(z,\infty)$ has exactly one critical point in each gap $(\mathbf{a}_i, \mathbf{b}_i)$. Let $\{\mathbf{d}_i\}$ be the collection of all critical points. E is called a regular Parreau-Widom set, if

$$\sum_i G_\Omega(\mathbf{d}_i, \infty) < \infty.$$

Widom [68] showed that in this case $H^2(\alpha) \neq \{0\}$ for all $\alpha \in \Gamma^*$. In general, (1.32) does not hold for arbitrary Parreau-Widom sets. This property is called *Direct Cauchy theorem* (DCT), see [30], and it is the crucial property of the domain such that the generalized Abel map is indeed bijective, see [59].

We want to give a description of the orthogonal complement of $H^2(\alpha)$ in

$$L^2(\alpha) = \left\{ f \in L^2(\mathbb{T}) \colon f \circ \gamma = e^{2\pi i \alpha(\gamma)} f, \ \gamma \in \Gamma \right\}.$$

Let

$$H_0^2(\mathbb{D}) = \{ f \in H^2(\mathbb{D}) : f(0) = 0 \} \quad \text{and} \quad H_0^2(\alpha) = H^2(\alpha) \cap H_0^2(\mathbb{D}).$$

For the standard Hardy spaces of the disc we know that

$$H^2(\mathbb{D})^\perp = \overline{H_0^2(\mathbb{D})}, \tag{1.33}$$

where the bar means complex conjugation, see [20, Section 7.2.]. In the following we will derive the character automorphic analog of this statement. The proofs will rely on properties of the averaging operator $P^\alpha : L^\infty \to L^\infty(\alpha)$ defined by

$$P^\alpha f = \frac{b}{b'} \sum_{\gamma \in \Gamma} e^{-2\pi i \alpha(\gamma)} \frac{\gamma'}{\gamma} f \circ \gamma,$$

see [51]. We list the most important properties of P^α:

- $\langle P^\alpha f, g \rangle = \langle f, g \rangle$, for all $f \in L^\infty$ and $g \in L^1(\alpha)$,
- $\|P^\alpha f\|_2 \leq \|f\|_2$, for all $f \in L^\infty$,
- $P^\alpha \mathcal{W} h \in H^\infty(\alpha)$, for all $h \in H^\infty$,
- $(P^\alpha \mathcal{W} h)(0) = \mathcal{W}(0) h(0)$ for all $h \in H^\infty$.

Lemma 1.35. *With the notation from above, the orthogonal complement of $H^2(\alpha)$ is given by*

$$H^2(\alpha)^\perp = \mathcal{W}\overline{H_0^2(\nu-\alpha)}.$$

Proof. Let $f \in L^2(\alpha)$ with $\langle g, f\rangle = 0$ for all $g \in H^2(\alpha)$. Then, by the properties of P^α, we obtain for all $h \in H^\infty$ that

$$0 = \langle f, P^\alpha \mathcal{W} h\rangle = \langle f, \mathcal{W}h\rangle = \langle \overline{\mathcal{W}}f, h\rangle.$$

Hence, by the density of H^∞ in H^2 and (1.33), we have $\overline{\mathcal{W}}f \in \overline{H_0^2(\alpha-\nu)}$, i.e., since $\mathcal{W}$ is an inner function $f \in \mathcal{W}\overline{H_0^2(\nu-\alpha)}$. For the other direction we use Lemma 1.33. Let $f = \mathcal{W}\overline{g}$ for $g \in H_0^2(\nu-\alpha)$. For all $h \in H^2(\alpha)$, we obtain

$$\langle f, h\rangle = \langle f, \mathcal{W}\overline{g}\rangle \int_{-\pi}^{\pi} \frac{f(\zeta)g(\zeta)}{\mathcal{W}(\zeta)}\frac{\mathrm{d}\zeta}{2\pi} = \left(\frac{fg}{\mathcal{W}}\right)(0) = 0.$$

□

Lemma 1.36. *Let $\theta \neq 1$ be a character automorphic inner function with character χ. Then*

$$\bigcap_{n=1}^{\infty}\left\{\theta^{-n}H^2(\alpha+n\chi)\right\}^\perp = \{0\}.$$

Proof. Let $f \in \bigcap_{n=1}^{\infty}\{\theta^{-n}H^2(\alpha+n\chi)\}^\perp$. That is, for all $n \in \mathbb{N}$ and all $h \in H^2(\alpha+n\chi)$ we have

$$0 = \langle f, \theta^{-n}h\rangle = \langle \theta^n f, h\rangle.$$

Therefore, by Lemma 1.35, we see that $\theta^n f \in H^2(\alpha+n\chi)^\perp = \mathcal{W}\overline{H_0^2(\nu-\alpha-n\chi)}$. Hence, there exists $g \in H_0^2(\nu-\alpha-n\chi)$ such that $\mathcal{W}\overline{f} = \theta^n g$, i.e.,

$$\mathcal{W}\overline{f} \in \bigcap_{n=1}^{\infty}\theta^n H_0^2(\nu-\alpha-n\chi).$$

For $g \in \bigcap_{n=1}^{\infty}\theta^n H_0^2(\nu-\alpha-n\chi)$ and arbitrary $\zeta_0 \in \mathbb{D}$, we see that

$$|g(\zeta_0)|^2 = \left|\langle g, \theta^n\overline{\theta^n(\zeta_0)}k_{\zeta_0}^{\nu-\alpha-n\chi}\rangle\right|^2 \le \|g\|^2|\theta(\zeta_0)|^{2n}k_{\zeta_0}^{\nu-\alpha-n\chi}(\zeta_0).$$

Since $k_{\zeta_0}^{\nu-\alpha-n\chi}(\zeta_0)$ is uniformly bounded in n from above, the right-hand side converges to zero as $n \to \infty$ and hence $g(\zeta_0) = 0$. □

Theorem 1.37 ([66])**.** *The system of functions*

$$e_n^\alpha(\zeta) = b^n(\zeta)\frac{k^{\alpha-n\mu}(\zeta)}{\sqrt{k^{\alpha-n\mu}(0)}}$$

(i) forms an orthonormal basis in $H^2(\alpha)$ for $n \in \mathbb{Z}_{\ge 0}$ and

(ii) forms an orthonormal basis in $L^2(\alpha)$ for $n \in \mathbb{Z}$.

Proof. $H^2(\alpha)$ can be decomposed into

$$H^2(\alpha) = \{e_0^\alpha\} \oplus H_0^2(\alpha), \tag{1.34}$$

where $H_0^2(\alpha) = bH^2(\alpha - \mu)$, iterating the previous step leads to

$$H^2(\alpha) = \{e_0^\alpha\} \oplus \left\{be_0^{\alpha-\mu}\right\} \oplus \left\{b^2 e_0^{\alpha-2\mu}\right\} \oplus \cdots .$$

It is easy to see that this system is complete. To prove (ii) note that $\{e_n^\alpha\}_{n=-N}^\infty = b^{-n}H^2(\alpha - n\mu)$. Hence, Lemma 1.36 implies that the system $\{e_n^\alpha\}_{n=-\infty}^\infty$ is complete. □

The map $\Gamma^* \longrightarrow J(E)$

First, we show that the multiplication with $\mathfrak{z}$ in the basis $\{e_n^\alpha\}$ is a Jacobi matrix $J(\alpha)$.

Theorem 1.38 ([66]). *The multiplication operator by $\mathfrak{z}$ in $L^2(\alpha)$ with respect to the basis $\{e_n^\alpha\}$ from Theorem 1.37 is the following Jacobi matrix $J = J(\alpha)$:*

$$\mathfrak{z}e_n^\alpha = a_n(\alpha)e_{n-1}^\alpha + b_n(\alpha)e_n^\alpha + a_{n+1}(\alpha)e_{n+1}^\alpha, \tag{1.35}$$

where

$$a_n(\alpha) = \mathcal{A}(\alpha - n\mu), \qquad \mathcal{A}(\alpha) = (\mathfrak{z}b)(0)\sqrt{\frac{k^\alpha(0)}{k^{\alpha+\mu}(0)}}$$

and

$$b_n(\alpha) = \mathcal{B}(\alpha - n\mu), \qquad \mathcal{B}(\alpha) = \frac{(\mathfrak{z}b)(0)}{b'(0)} + \left\{\frac{(k^\alpha)'(0)}{k^\alpha(0)} - \frac{(k^{\alpha+\mu})'(0)}{k^{\alpha+\mu}(0)}\right\} + \frac{(\mathfrak{z}b)'(0)}{b'(0)}.$$

Proof. Since $\mathfrak{z}bk^{\alpha-\mu} \in H^2(\alpha)$ and $\mathfrak{z}$ is a self-adjoint operator, we obtain that

$$\mathfrak{z}bk^{\alpha-\mu} = c_0k^\alpha + c_1bk^{\alpha-\mu} + c_2b^2k^{\alpha-2\mu}. \tag{1.36}$$

Expanding (1.36) at the origin yields

$$\left\{(\mathfrak{z}b)(0)k^{\alpha-\mu}(0) - c_0k^\alpha(0)\right\}$$
$$+ \left\{(\mathfrak{z}b)'(0)k^{\alpha-\mu}(0) + (\mathfrak{z}b)(0)(k^{\alpha-\mu})'(0) - c_0(k^\alpha)'(0) - c_1b'(0)k^{\alpha-\mu}(0)\right\}\zeta + \cdots = 0.$$

□

It is convenient to introduce the dual basis $\{\tilde{e}_n^\alpha\}$

$$\tilde{e}_n^\alpha = \mathcal{W}\overline{be_{-n-1}^\alpha}, \quad \text{on } \mathbb{T} \tag{1.37}$$

of the dual Hardy space

$$b\tilde{H}^2(\nu - \alpha - \mu) := \mathcal{W}\overline{L^2(\alpha) \ominus H^2(\alpha)}.$$

That is, if $\tau J = J(\tilde{H}^2(\nu - \alpha - \mu))$, then the entries of τJ are given by

$$\tau a_n = a_{-n}(\alpha), \quad \tau b_n = b_{-n-1}(\alpha).$$

Theorem 1.39. *Let $\alpha \in \Gamma^*$ and $J(\alpha)$ be defined as in Theorem 1.38. Then the corresponding resolvent functions r_+^α and r_-^α are given by*

$$r_+^\alpha \circ \mathfrak{z} = -\frac{e_0^\alpha}{a_0(\alpha)e_{-1}^\alpha} \quad \text{and} \quad r_-^\alpha \circ \mathfrak{z} = -\frac{\tilde{e}_0^\alpha}{a_0(\alpha)\tilde{e}_{-1}^\alpha}.$$

In particular, $J(\alpha) \in J(E)$.

Proof. Since multiplication by $\mathfrak{z}$ is the model operator for $J(\alpha)$, its spectrum is the set E. By (1.25), we see that

$$r_+^\alpha \circ \mathfrak{z} = \frac{-1}{\mathfrak{z} - b_0(\alpha) + a_1^2(\alpha) r_+^{\alpha-\mu} \circ \mathfrak{z}}.$$

Let us introduce for a second

$$m_+^\alpha \circ \mathfrak{z} = -\frac{e_0^\alpha}{a_0(\alpha)e_{-1}^\alpha}.$$

(1.35) yields

$$\begin{aligned} -\frac{1}{m^\alpha \circ \mathfrak{z}} &= \frac{a_0(\alpha)e_{-1}^\alpha}{e_0^\alpha} \\ &= \mathfrak{z} - b_0(\alpha) - a_1(\alpha)^2 \frac{e_1^\alpha}{a_1(\alpha)e_0^\alpha} \\ &= \mathfrak{z} - b_0(\alpha) + a_1(\alpha)^2 m_+^{\alpha-\mu} \circ \mathfrak{z}. \end{aligned}$$

Iterating this continued fraction expansions shows that $r_+^\alpha = m_+^\alpha$. Applying the same for τJ shows the second identity. By (1.37), we see that $J(\alpha)$ is reflectionless on E. □

The map $\Gamma^* \longleftrightarrow D(E)$

For finitely connected domains Widom [67, Theorem 7.1] already proved that the map is one-to-one.

Theorem 1.40. *Let $\alpha \in \Gamma^*$ and $\mathbf{d}_i$ be the critical points of $G_\Omega(z, \infty)$ and $e_0^\alpha e_0^\alpha(0) = k^\alpha$. Then there exists a unique divisor $\{\mathbf{x}_i, \epsilon_i\} \in D(E)$, such that*

$$e^\alpha(\zeta) = \prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)^{\frac{1+\epsilon_i}{2}} \sqrt{\frac{\mathcal{W}(\zeta)}{\prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)} \prod_{i=1}^g \frac{\mathfrak{z}(\zeta) - \mathbf{x}_i}{\mathfrak{z}(\zeta) - \mathbf{d}_i}}. \tag{1.38}$$

Moreover, every $D \in D(E)$ defines a reproducing kernel for a certain Hardy space $H^2(\alpha)$.

Proof. We only need to prove that the map is also onto. Let $\{\mathbf{x}_i, \epsilon_i\} \in D(E)$. To this we associate the reproducing kernel

$$e_0^\alpha(\zeta) = \prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)^{\frac{1+\epsilon_i}{2}} \sqrt{\frac{\mathcal{W}(\zeta)}{\prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)} \prod_{i=1}^g \frac{\mathfrak{z}(\zeta) - \mathbf{x}_i}{\mathfrak{z}(\zeta) - \mathbf{d}_i}}.$$

Note that by our normalization

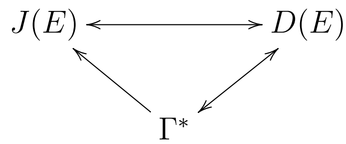
$$e_0^\alpha(0) > 0.$$

Using that $\mathfrak{z}$ is real on $\mathbb{T}$ and (1.32), we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} f^\alpha(\zeta)\overline{e_0^\alpha(0) e_0^\alpha(\zeta)} \frac{\mathrm{d}\zeta}{2\pi} \\
=&e_0^\alpha(0) \int_{-\pi}^{\pi} \frac{f^\alpha(\zeta)\prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)^{\frac{1-\epsilon_i}{2}}}{\mathcal{W}(\zeta)} \sqrt{\frac{\mathcal{W}(\zeta)}{\prod_{i=1}^g b_{\mathbf{x}_i}(\zeta)} \prod_{i=1}^g \frac{\mathfrak{z}(\zeta)-\mathbf{x}_i}{\mathfrak{z}(\zeta)-\mathbf{d}_i}} \frac{\mathrm{d}\zeta}{2\pi} \\
=&e_0^\alpha(0) f^\alpha(0) \prod_{i=1}^g b_{\mathbf{x}_i}(0)^{-\frac{\epsilon_i}{2}} \mathcal{W}(0)^{-\frac{1}{2}} \left(\prod_{i=1}^g \frac{\mathfrak{z}(\zeta)-\mathbf{x}_i}{\mathfrak{z}(\zeta)-\mathbf{d}_i}\right)(0) = f^\alpha(0).
\end{aligned}$$

□

Hence, we have the following diagram



Thus, we can define the Abel map $\tilde{\mathcal{A}} : J(E) \to \Gamma^*$, such that for all $J \in J(E)$ we have $J = J(\tilde{A}(J))$.

Remark 1.41. (i) For a parametrization of the isospectral set of Jacobi matrices associated to finite gap sets (as discussed in the previous chapter) in terms of theta functions on the associated Riemann surfaces see [6, 9, 36], see also [61].

(ii) As already mentioned, the approach presented above was in fact developed to deal with more complicated sets such as Parreau-Widom sets. In addition to bijectivity of the generalized Abel map, DCT also implies that $\mathcal{A}, \mathcal{B}$ are continuous functions on $\mathbb{R}^\infty/\mathbb{Z}^\infty$ and hence Theorem 1.38 shows that all elements $J(\alpha)$ of $J(E)$ are almost periodic.

Finally, we would like to go back to the periodic case and use the functional model to prove a remarkable characterization of the isospectral torus of periodic Jacobi matrices, which is called the *magic formula*, see [18].

Theorem 1.42. *Let J be a p-periodic Jacobi matrix with discriminant Δ_p^J and spectrum E. Then for every two-sided Jacobi matrix we have*

$$\tilde{J} \in J(E) \iff \Delta_p^J(\tilde{J}) = S^p + S^{-p}.$$

Proof. Let P_p be the associated normalized Chebyshev polynomial. Then we know that $\Delta_p^J = 2P_p$. Moreover, we have

$$b(\zeta) = B(\mathfrak{z}(\zeta), w), \quad (\mathfrak{z}, w) \in \mathcal{R}_+.$$

Hence, (1.9) shows that

$$\Delta_p^J(\mathfrak{z}(\zeta)) = b(\zeta)^p + \frac{1}{b(\zeta)^p}. \tag{1.39}$$

Given $\tilde{J} \in J(E)$, then $\tilde{J} = J(\alpha)$, where $\alpha = \tilde{A}(\tilde{J})$. Thus, (1.39) is just the magic formula in the corresponding functional model. Note that multiplication by $\mathfrak{z}$ is the symbol of J and b the symbol of the shift S. Vice versa, let $\tilde{J}$ satisfy the magic formula. Then Naĭman's lemma (cf. [56, Lemma 8.2.4]) implies that $\tilde{J}$ is periodic. Hence, since $\tilde{J} \in J(\sigma(\tilde{J}))$, we see that

$$\Delta_p^{\tilde{J}}(\tilde{J}) = S^p + S^{-p} = \Delta_p^{J}(\tilde{J}).$$

Note that for a polynomial, P, and a Jacobi matrix, J, by considering the most outer non vanishing diagonal of $P(J)$, we see that $P(J) = 0$ already implies that $P \equiv 0$. Hence, we obtain that $\Delta_p^{\tilde{J}} = \Delta_p^{J}$ and we have $\tilde{J} \in J(E)$. □

Chapter 2

Main Results

2.1 Periodic GMP matrices

According to Corollary 1.28, if

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j) \tag{2.1}$$

is the spectrum of a periodic Jacobi matrix, the coefficients of the Jacobi matrices can be used as coordinates for the real subtorus $J(E)$. Generally an algebraic description of the Jacobian, $\mathrm{Jac}(\mathcal{R})$, is known, but it is not given in terms of coefficient sequences. Mumford [46] represents the affine part of $\mathrm{Jac}(\mathcal{R})$ by means of coefficients of certain special polynomials. We describe this construction for the real part of $\mathrm{Jac}(\mathcal{R})$: Given $D = \{\mathbf{x}_i, \epsilon_i\} \in D(E)$ and the shifted divisor $D^{(1)} = \{\mathbf{x}_i^{(1)}, \epsilon_i^{(1)}\}$ these polynomials are U, W and V, where

$$U(z) = \prod_{i=1}^{g} (z - \mathbf{x}_i), \quad W(z) = \prod_{i=1}^{g} (z - \mathbf{x}_i^{(1)})$$

and V is defined by

$$a_0^2 r_+^J(z) = \frac{-R_E(z) + V(z)}{U(z)}, \quad -\frac{1}{r_-^J(z)} = \frac{-R_E(z) - V(z)}{U(z)},$$

see also (4.35). Note that by (1.23) and (1.24) this yields the factorization

$$-R_E(z)^2 + V(z)^2 = -a_0^2 U(z) W(z),$$

which was used by Mumford as defining equations for the polynomials.

GMP matrices are a new class of operators such that for arbitrary systems of intervals the isospectral torus consists always of periodic operators and the coefficients can always be used as coordinates for the isospectral torus. In particular, one obtains a magic formula type characterization of the isospectral torus of GMP matrices.

The magic formula was crucial in proving the first generalization of the remarkable Killip–Simon theorem [33]. Let J_+ be a one-sided Jacobi matrix with spectral measure $d\sigma_+ = \sigma_+'(x)dx + d\sigma_s$ and J_+^0 the free discrete Schrödinger operator from Example 1.26 with spectral measure $d\sigma_0(x) = \chi_{[-2,2]}(x)\sqrt{4-x^2}dx$. Killip and Simon showed that the condition on the operator (op) $J_+ - J_+^0$ is of Hilbert-Schmidt class

is satisfied if and only if the measure $d\sigma_+$ is supported on $[-2,2] \cup X$ and satisfies the spectral condition

$$\text{(sp)} \quad \int_{-2}^{2} |\log \sigma'(x)| \sqrt{4-x^2}\mathrm{d}x + \sum_{x_k \in X} \sqrt{x_k^2 - 4}^3 < \infty.$$

The spectral condition in particular implies that the absolutely continuous part of $d\sigma_+$ must not vanish almost everywhere on $[-2,2]$, but gives no restriction on the singular part on E. Recalling the von Neumann theorem, highlights how surprising the above mentioned statement is. It states that for every self-adjoint operator there exists a self-adjoint Hilbert Schmidt perturbation which completely destroys the a.c. part of the spectrum. Only the information about the special structure (namely only Jacobi matrices of Hilbert Schmidt class) of the perturbation gives that the a.c. part of the spectrum of J_+^0 is completely preserved.

Using the magic formula characterization, Killip and Simon, jointly with Damanik were able to generalize this result to systems of intervals E, which are spectra of periodic Jacobi matrices such that $|\Delta_p^J(c)| > 2$ for all critical points c. They found that

(opp) $\Delta_p^J(J_+) - (S_+^p + (S_+^*)^p)$ is of Hilbert Schmidt class

if and only if $d\sigma_+$ is supported on $E \cup X$ and satisfies

$$\text{(spp)} \quad \int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R}\setminus E)}\mathrm{d}x + \sum_{x_k \in X} \sqrt{\mathrm{dist(x_k, E)}}^3 < \infty.$$

Note that $S_+^p + (S_+^*)^p$ is the matrix analog of J_+^0 were $a_n = 1, b_n = 0$ is substituted by the $p \times p$ identity- and zero-matrix. A hard part was to extract from the very involved operator condition (opp) an explicit condition on the coefficients. Let $\text{dist}(J_+^1, J_+^2)$ be any distant inducing the weak topology of pointwise convergence, e.g.,

$$\text{dist}(J_+^1, J_+^2) := \sum_{n \in \mathbb{N}} 2^{-n}(|a_n^1 - a_n^2| + |b_n^1 - b_n^2|),$$

where J_+^1 and J_+^2 are defined by the coefficient sequences $\{a_n^1, b_n^1\}_{n\in\mathbb{N}}$ and $\{a_n^2, b_n^2\}_{n\in\mathbb{N}}$, respectively. Moreover, let us define the distant to the torus in the usual way by

$$\text{dist}(J_+, J(E)) := \inf\{\text{dist}(J_+, P_+ \mathring{J} P_+^*) : \quad \mathring{J} \in J(E)\}.$$

Then they showed that (opp) is equivalent to

$$\sum_{n \in \mathbb{N}} \text{dist}((S_+^*)^n J_+ S_+^n, J(E))^2 < \infty. \tag{2.2}$$

The idea of GMP matrices is to substitute the polynomial, Δ_p^J, by a rational function. In [23], we carried out this idea for the simplest case, namely if E is an arbitrary union of two distinct intervals. Note that the Damanik, Killip and Simon theorem only covers two intervals of equal length. Nevertheless, we can always find a rational function, Δ_E, of the form

$$\Delta_E(z) = \lambda_0 z + \mathbf{c}_0 + \frac{\lambda_1}{\mathbf{c}_1 - z}, \qquad \lambda_0, \lambda_1 > 0, \tag{2.3}$$

such that $\Delta_E^{-1}([-2,2]) = E$. By a linear change of variable we may assume that $\mathbf{c}_1 = 0$. This suggests to consider matrices obtained by orthonormalizing the family of functions

$$1,\ -\frac{1}{x},\ x,\ \frac{(-1)^2}{x^2},\ \dots, \tag{2.4}$$

for a given real compactly supported measure $d\sigma_+$. Denoting this basis by φ_n, we call the matrix of multiplication by the independent variable with respect to this basis a one-sided SMP matrix (abbreviation for Strong Moment Problem). Let us mention that they are also called Jacobi-Laurent matrices; cf. [32].

The connection to CMV matrices should not go unmentioned. CMV matrices are the Jacobi matrix analog for measures supported on the unit circle. Already Szegő discussed orthogonal polynomials, ψ_n, with respect to a measure, $d\sigma$, supported on the unit circle and showed that there are $\{\alpha_n\}_{n=0}^{\infty}$ in $\mathbb{D}$, called Verblunsky coefficients, so that

$$\sqrt{1-|\alpha_n|^2}\psi_{n+1}(z) = z\psi_n(z) - \overline{\alpha_n} z^n \overline{\psi_n(1/\overline{z})}.$$

Due to Verblunsky [65], who defined them in another context, the map $d\sigma \mapsto \{\alpha_n\}$ is one-to-one and onto all of $\mathbb{D}^\infty$. Recent developments are due to Cantero, Moral and Velázques [15]. They considered bases obtained by orthonormalizing families of the sort (2.4) and showed that the matrix of the multiplication operator is a special structured five-diagonal matrix. For a given measure, $d\sigma$, the entries can be given in terms of the Verblunsky coefficients. Recognizing this characteristic structure, they could use it to give a constructive definition of CMV matrices, which uniquely defined them, in the sense that there is a one-to-one correspondence between measures and CMV matrices.

In [23], we were also able to identify this characteristic structure for SMP matrices and to give a constructive definition of them. Again, it was then more convenient for us to define them as two-sided matrices. Roughly speaking, a SMP matrix A and its shifted inverse $-S^{-1}A^{-1}S$ (note that we assumed that 0 is not in the spectrum) are five-diagonal matrices such that all even entries on the most outer diagonal vanish and the odd ones are positive. This structure perfectly fits to the following "generalized magic formula":

Proposition 2.1 ([23]). *Let E be an arbitrary union of two intervals around zero and Δ_E the corresponding rational function of (2.3). Moreover, let $A(E)$ be the set of all two-periodic SMP matrices with its spectrum on E. Then*

$$A \in A(E) \iff \Delta_E(A) = S^2 + S^{-2}.$$

To deal with arbitrary finite gap sets of the form (2.1), one first has to generalize (2.3), which is done by the following lemma.

The Ahlfors function, Ψ, of the domain $\overline{\mathbb{C}} \setminus E$ is the function that is bounded by one in modulus in Ω, vanishes at infinity and maximizes the "derivative" at infinity, i.e., the value $|z\Psi(z)|(\infty)$, in this class. The extremal value, $\mathrm{Cap}_a(E)$, is called *analytic capacity*.

Lemma 2.2. *The function*

$$\Delta_E(z) := \Psi(z) + \frac{1}{\Psi(z)} \tag{2.5}$$

is a rational function of the form

$$\Delta_E(z) = \lambda_0 z + \mathbf{c}_0 + \sum_{j=1}^{g} \frac{\lambda_j}{\mathbf{c}_j - z}, \tag{2.6}$$

with $\lambda_j > 0$, $j \geq 0$, $\mathbf{c}_j \in (\mathbf{a}_j, \mathbf{b}_j)$, $j \geq 1$ *and*

$$E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j) = \Delta_E^{-1}([-2, 2]). \tag{2.7}$$

In fact, if we demand that $\operatorname{Im} \Delta_E(z) > 0$ *for* $\operatorname{Im} z > 0$ *and* $\lim_{z\to\infty} \Delta_E(z) = \infty$, Δ_E *is the unique rational function with the property* (2.7).

Proof. Due to [50], we have

$$\frac{1 - \Psi(z)}{1 + \Psi(z)} = \sqrt{\prod_{j=0}^{g} \frac{z - \mathbf{a}_j}{z - \mathbf{b}_j}} =: G(z). \tag{2.8}$$

Therefore,

$$\Delta_E(z) := \Psi(z) + \frac{1}{\Psi(z)} = 2\frac{1 + G^2(z)}{1 - G^2(z)},$$

is of the form $P_{g+1}(z)/Q_m(z)$, where $m \leq g$. Like in [38, Chapter VII], we see that $\operatorname{Im} G^2(z) > 0$ for $\operatorname{Im} z > 0$, which then clearly also holds for Δ_E. Therefore, G^2 is increasing on the interval $(\mathbf{a}_j, \mathbf{b}_j)$ and has a zero at $\mathbf{a}_j$ and a pole at $\mathbf{b}_j$. Hence, there is exactly one pole of Δ_E in each gap. Since $G^2(\infty) = 1$, there is also a pole at infinity. To prove uniqueness let $\Delta(z)$ be a rational function with the claimed properties. First, we notice that, due to the argument principle, it is not possible that Δ has a pole in the upper half plane. This and $\lim_{z\to\infty} \Delta(z) = \infty$ already implies that Δ is of the form (2.6). Since $\Delta'(x) > 0$ on $\mathbb{R} \setminus \{\mathbf{c}_1, \dots, \mathbf{c}_g\}$, $\Delta^{-1}([-2, 2]) = E$ implies

$$\Delta(\mathbf{a}_j) = 2 \qquad \text{and} \qquad \Delta(\mathbf{b}_j) = -2 \qquad \text{for} \quad j \geq 0.$$

This defines Δ uniquely. □

Let $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_g\}$ be a collection of distinct real points and $d\sigma_-$ a measure such that the points $\mathbf{c}_k$ don't belong to its support. Like SMP matrices, we define a one-sided GMP matrix, A_-, as the matrix of the multiplication operator with respect to the basis obtained by orthonormalizing the family of functions

$$1, \frac{1}{\mathbf{c}_g - x}, \frac{1}{\mathbf{c}_{g-1} - x}, \dots, \frac{1}{\mathbf{c}_1 - x}, x, \frac{1}{(\mathbf{c}_g - x)^2}, \dots.$$

This is discussed in detail in the Appendix of [72]; see also [11]. Their characteristic structure, which looks quite complicated at the first glance, will be used in the following definition, but first we would like to point out another property of multiplication operators with respect to rational functions. Namely, since $\mathbf{c}_k$ does not belong to the support of the measure, we can also consider multiplication by $\frac{1}{\mathbf{c}_k - x}$. Hence, if the first block of A_- corresponds to the basis

$$\left[1, \frac{1}{\mathbf{c}_1 - x}, \dots, \frac{1}{\mathbf{c}_g - x}\right] \tag{2.9}$$

then the linear change of variable $y = \frac{1}{\mathbf{c}_k - x}$ leads to the basis related to

$$\left[-\frac{1}{y}, \frac{1}{y(\mathbf{c}_1) - y}, \dots, 1, \dots, \frac{1}{y(\mathbf{c}_g) - y}\right], \tag{2.10}$$

which says that the shifted resolvents should be of the same shape. In fact, in the construction the spaces (2.9) and (2.10) serve as cyclic subspaces for $\Delta(x)$ and $\tilde{\Delta}(y) = \Delta(x)$, respectively. See also the proof of Theorem 2.5. The structure of A_- and this certain invariance property of the resolvents is now used as a definition for two-sided GMP matrices.

By T^* we denote the conjugated operator to an operator T, or the conjugated matrix if T is a matrix. In particular, for a column vector $\vec{p} \in \mathbb{C}^{g+1}$, $(\vec{p})^*$ is a $(g+1)$-dimensional row vector. The notation T^- denotes the upper triangular part of a matrix T (excluding the main diagonal), and $T^+ = T - T^-$ is its lower triangular part (including the main diagonal).

First of all, the GMP class depends on an ordered collection of distinct points $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_g\}$.

Definition 2.3. We say that A is of the class $\mathbb{A}$ if it is a $(g+1)$-block Jacobi matrix

$$A = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & A^*(\vec{p}_{-1}) & B(\vec{\mathbf{p}}_{-1}) & A(\vec{p}_0) & & \\ & & A^*(\vec{p}_0) & B(\vec{\mathbf{p}}_0) & A(\vec{p}_1) & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

such that

$$\vec{\mathbf{p}} = (\vec{p}, \vec{q}) \in \mathbb{R}^{2g+2}, \qquad A(\vec{p}) = \delta_g \vec{p}^*, \qquad B(\vec{\mathbf{p}}) = (\vec{q}\vec{p}^*)^- + (\vec{p}\vec{q}^*)^+ + \tilde{\mathbf{C}},$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{c}_1 & & & \\ & \ddots & & \\ & & \mathbf{c}_g & \\ & & & 0 \end{bmatrix}, \qquad \vec{p}_j = \begin{bmatrix} p_0^{(j)} \\ \vdots \\ p_g^{(j)} \end{bmatrix}, \qquad \vec{q}_j = \begin{bmatrix} q_0^{(j)} \\ \vdots \\ q_g^{(j)} \end{bmatrix}, \qquad p_g^{(j)} > 0.$$

We call $\{\vec{\mathbf{p}}_j\}_{j\in\mathbb{Z}}$ the generating coefficient sequences (for the given A).

Definition 2.4. A matrix $A \in \mathbb{A}$ belongs to the GMP class if the matrices $\{\mathbf{c}_k - A\}_{k=1}^g$, for $1 \le k \le g$, are invertible, and moreover $S^{-k}(\mathbf{c}_k - A)^{-1}S^k$ are also of the class $\mathbb{A}$. To abbreviate we write $A \in \mathrm{GMP}(\mathbf{C})$.

We call a GMP matrix one-block periodic or simply periodic if $\vec{\mathbf{p}}_j = \vec{\mathbf{p}}$ for all $j \in \mathbb{Z}$. Thus, we pay a quite high price in giving up the simple structure of Jacobi matrices, but in return we get (2.5), which will in particular allow us to parametrize the finite gap class of almost periodic Jacobi matrices by periodic GMP matrices.

We would like to explain the idea of periodic GMP matrices by means of the corresponding functional models. Recall that $b(\zeta)$ denotes the Blaschke product related to ∞ in Ω; cf. Section 1.2.4. Then periodicity of the elements of the isospectral torus associated to E reduces to the question if there exists $p \in \mathbb{N}$ such that the function $\Phi(\zeta) = b(\zeta)^p$ has a trivial character. The idea of GMP matrices is to substitute Φ by the Ahlfors function Ψ, which is by definition single-valued. In the same way as for Jacobi matrices, cf. (1.34), the decomposition

$$H^2(\alpha) = K_\Psi(\alpha) \oplus \Psi H^2(\alpha),$$

leads to a new basis $\{f_n^\alpha\}_{n=-\infty}^\infty$ such that $f_{n+p}^\alpha = \Psi f_n^\alpha$, $K_\Psi(\alpha) = \{f_0^\alpha, \dots, f_g^\alpha\}$ and the multiplication by $\mathfrak{z}$ is a GMP matrix. Note that the property $f_{n+p}^\alpha = \Psi f_n^\alpha$ shows that the corresponding matrix is periodic. Defining the isospectral torus of GMP matrices by

$$A(E, \mathbf{C}) = \{A \in \mathrm{GMP}(\mathbf{C})\colon \sigma(A) = E \text{ and } A \text{ is periodic}\},$$

we obtain the following analogue of the Abel map for GMP matrices.

Theorem 2.5. *Let E be a finite gap set and $\mathbf{C}$ be a fixed ordering of the zeros of the corresponding Ahlfors function. Then*

$$A(E, \mathbf{C}) = \left\{A(\alpha, \mathbf{C})\colon \alpha \in \mathbb{R}^g/\mathbb{Z}^g\right\},$$

where $A(\alpha, \mathbf{C})$ is the multiplication by the independent variable with respect to the basis $\{f_n^\alpha\}$. The map $\alpha \mapsto A(\alpha, \mathbf{C})$ is one-to-one and onto up to the identification $(p_j, q_j) \mapsto (-p_j, -q_j)$ for $j = 0, \dots, g-1$.

Moreover, (2.5) is the magic formula for GMP matrices in terms of our functional model.

Theorem 2.6. *Let $A \in \mathrm{GMP}(\mathbf{C})$. Then*

$$A \in A(E, \mathbf{C}) \iff \Delta_E(A) = S^{g+1} + S^{-(g+1)}.$$

This was one of the main observations which allowed Yuditskii in [72] to generalize the Killip–Simon theorem to arbitrary systems of intervals. The ideas already introduced by Damanik, Killip and Simon led to a Killip-Simon theorem for GMP matrices. Let $A \in \mathrm{GMP}(\mathbf{C})$ be a two-sided GMP matrix with associated matrix spectral measure $\mathrm{d}\Sigma$. Then

(opg) $\Delta_E(A) - (S^{g+1} + S^{-(g+1)})$ is of Hilbert Schmidt class if and only if $\mathrm{d}\Sigma$ is supported on $E \cup X$ and

$$\text{(spg)} \quad \int_E |\log\det\Sigma'(x)|\sqrt{\operatorname{dist}(x,\mathbb{R}\setminus E)}\mathrm{d}x + \sum_{x_k\in X}\sqrt{\operatorname{dist}(\mathrm{x_k},\mathrm{E})}^3 < \infty.$$

Nevertheless, the final goal is to obtain the corresponding result for Jacobi matrices. This is accomplished in the following way: The vectors e_{-1} and $\tilde{e}_0 = \frac{1}{\|\vec{p}_0\|}P_+Ae_{-1}$, form a cyclic subspace for the GMP matrix A. Moreover, to every Jacobi matrix, J, there exists a unique GMP matrix (up to a certain identification; cf. Theorem 2.5) A and a unitary operator $U:\ell^2\to\ell^2$ such that $J = U^*AU$, $Ue_{-1} = e_{-1}$ and $Ue_0 = \tilde{e}_0$. Hence, the entries of J and A are related by

$$a_0 = \|\vec{p}_0\|, \quad b_0 = p_g^{(-1)}q_g^{(-1)}. \tag{2.11}$$

Since the operators are unitarily equivalent, the spectral conditions are the same. Thus, it remains to translate (*opg*) into a condition on J_+. To this end, Yuditskii found a new integrable system, which he called *Jacobi flow* on GMP matrices. It is defined as the discrete dynamical system on the set of GMP matrices that conjugates the shift on Jacobi matrices. By means of this flow, one can use (2.11) to iteratively read out the coefficients of the corresponding Jacobi matrix. Since $S^{-1}J(\alpha)S = J(\alpha-\mu)$, it is clear how the Jacobi flow should be defined on $A(E,\mathbf{C})$.

Definition 2.7. We define the Jacobi flow on $A(E,\mathbf{C})$ as the dynamical system generated by the following map:

$$\mathcal{J}A(\alpha) = A(\alpha-\mu), \qquad \alpha\in\Gamma^*.$$

For a parametric description of the Jacobi flow for general non-periodic GMP matrices see [72, Theorem 4.5]. The new notions of GMP matrices and the Jacobi flow on them allowed Yuditskii to generalize the Killip-Simon theorem to arbitrary systems of intervals. Moreover, he gave a very explicit translation of the condition (2.2) in terms of the functions $\mathcal{A}$ and $\mathcal{B}$ on Γ^* defined in Theorem 1.38.

Theorem 2.8 ([72])**.** *Let J_+ be a one-sided Jacobi matrix with $\mathrm{d}\sigma_+ = \sigma'_+(x)\mathrm{d}x + \mathrm{d}\sigma_s$. J_+ satisfies the spectral condition (spp) if and only if there exist $\epsilon_\alpha(n)\in\ell^2_+(\mathbb{R}^g)$ and $\epsilon_a(n)\in\ell^2_+$, $\epsilon_b(n)\in\ell^2_+$ such that*

$$a(n)^2 = \mathcal{A}\Big(\sum_{k=0}^n\epsilon_\alpha(k)-\mu n\Big)+\epsilon_a(n), \quad b(n) = \mathcal{B}\Big(\sum_{k=0}^n\epsilon_\alpha(k)-\mu n\Big)+\epsilon_b(n).$$

Finally, we would like to point out that the direct spectral theory of periodic GMP matrices has numerous similarities to the one of periodic Jacobi matrices. It is based on the fact that like Jacobi matrices GMP matrices can be written as a two-dimensional perturbation of a block diagonal matrix, i.e., we have

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+\end{bmatrix} + a_0(\langle\cdot,e_{-1}\rangle\tilde{e}_0 + \langle\cdot,\tilde{e}_0\rangle e_{-1}), \quad a_0 = \|\vec{p}_0\|.$$

Definition 2.9. Let $A \in \mathrm{GMP}(\mathbf{C})$ be a periodic GMP matrix with coefficients $\vec{p}$ and $\vec{q}$. Let $\mathbf{p} = \begin{bmatrix} p & q \end{bmatrix} \in \mathbb{R}^2$. We introduce the matrix functions

$$\mathfrak{a}(z;\mathbf{p}) = \mathfrak{a}(z,\infty;p,q) = \begin{bmatrix} 0 & -p \\ \frac{1}{p} & \frac{z-pq}{p} \end{bmatrix}, \tag{2.12}$$

and

$$\mathfrak{a}(z,\mathbf{c};\mathbf{p}) = I - \frac{1}{\mathbf{c}-z}\begin{bmatrix} p \\ q \end{bmatrix}\begin{bmatrix} p & q \end{bmatrix}\mathfrak{j}, \qquad \mathfrak{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Like in the case of Jacobi matrices, we call the product

$$\mathfrak{A}(z) = \mathfrak{a}(z,\mathbf{c}_1;\mathbf{p}_0)\mathfrak{a}(z,\mathbf{c}_2;\mathbf{p}_1)\cdots\mathfrak{a}(z,\mathbf{c}_g;\mathbf{p}_{g-1})\mathfrak{a}(z;\mathbf{p}_g)$$

transfer matrix associated with the given A. Moreover, we define its discriminant by

$$\Delta^A(z) = \operatorname{tr}\mathfrak{A}(z). \tag{2.13}$$

Theorem 2.10. *Let $A \in \mathrm{GMP}(\mathbf{C})$ be a periodic GMP matrix with coefficients $\vec{p}$ and $\vec{q}$. Then A has purely absolutely continuous spectrum, which is given by*

$$\sigma(A) = \sigma_{ac}(A) = \left\{z \in \mathbb{C}\colon \Delta^A(z) \in [-2,2]\right\}.$$

If for a given set E, we take an element $A \in A(E,\mathbf{C})$ of the isospectral torus, then we show in Lemma 3.12 that indeed $\Delta_E = \Delta^A$. This allows us to explain an alternative definition of GMP matrices. We define for a periodic GMP matrix with generating coefficients $\vec{p}$ and $\vec{q}$

$$\Lambda_k(\vec{\mathbf{p}}) = -\operatorname{tr}\left\{\prod_{m=0}^{k-2}\mathfrak{a}(\mathbf{c}_k,\mathbf{c}_{m+1};\mathbf{p}_m)\mathbf{p}_{k-1}\mathbf{p}_{k-1}^*\mathfrak{j}\prod_{m=k}^{g-1}\mathfrak{a}(\mathbf{c}_k,\mathbf{c}_{m+1};\mathbf{p}_m)\mathfrak{a}(\mathbf{c}_k;\mathbf{p}_g)\right\}, \tag{2.14}$$

where by definition $\Lambda_k(\vec{\mathbf{p}}) = -\operatorname{Res}_{\mathbf{c}_k}\operatorname{tr}\mathfrak{A}(z)$.

Let us consider the last non-zero diagonal of $\Delta_E(A)$, i.e., $\Delta_E(A)_{j,g+1+j}$ for $0 \le j \le g$. By definition of GMP matrices, $(\mathbf{c}_k - A)^{-1}_{k-1,g+k} > 0$ for $1 \le k \le g$, whereas $(\mathbf{c}_l - A)^{-1}_{k-1,g+k} = A_{k-1,g+k} = 0$ for $l \neq k$. Moreover, $(\mathbf{c}_k - A)^{-1}_{g,2g+1} = 0$ for $1 \le k \le g$ and $A_{g,2g+1} > 0$. Thus, on the last non-zero diagonal of $\Delta_E(A)$ only one of the summands is non-zero. Note that the relation $\Delta_E(z) = \Delta^A(z)$ implies $\lambda_k = \Lambda_k(\vec{\mathbf{p}})$. The previous consideration, the magic formula and this identity yield

$$\Lambda_k(\vec{\mathbf{p}})(\mathbf{c}_k - A)^{-1}_{k-1,g+k} = 1 \qquad \text{for} \quad 1 \le k \le g.$$

It is important to mention that all this just served as explanation, but is not necessary to prove the following alternative definition of GMP matrices.

Theorem 2.11. *Let $A \in \mathbb{A}$ be periodic with generating coefficients $\vec{\mathbf{p}}$. Then $A \in \mathrm{GMP}(\mathbf{C})$ if and only if*

$$\Lambda_k(\vec{\mathbf{p}}) > 0 \qquad \textit{for} \quad 1 \le k \le g.$$

The proof is based on the idea that one can find the entries of the inverse matrices $(\mathbf{c}_k - A)^{-1}$ explicitly; cf. [72, Lemma 3.2 and Theorem 3.3]

The relation $\Delta_E(z) = \Delta^A(z)$ finally leads to an algebraic description for $A(E, \mathbf{C})$.

Theorem 2.12. *Let $A \in \mathrm{GMP}(\mathbf{C})$ be periodic with generating coefficients $\vec{\mathbf{p}}$. Then $A \in A(E, \mathbf{C})$ if and only if*

$$p_g = \frac{1}{\lambda_0}, \quad q_g = -\mathbf{c}_0 - \lambda_0 \sum_{j=1}^{g-1} p_j q_j, \qquad \Lambda_k(\vec{\mathbf{p}}) = \lambda_k \qquad \textit{for} \quad k = 1, \dots, g, \tag{2.15}$$

where $\Lambda_k(\vec{\mathbf{p}})$ is defined as in (2.14).

2.2 Chebyshev and polynomial Ahlfors problems

2.2.1 Szegő Widom Asymptotic of Circular Arcs

Let us recall the results of Widom on asymptotics of Chebyshev polynomials. If E is a finite union of Jordan regions, he proved that

$$\lim_{n\to\infty} \frac{\|T_n\|_E}{\|w_{n\mu}\|_\infty \operatorname{Cap}(E)^n} = 1,$$

where $w_{n\mu}$ is the solution of the character automorphic H^∞ problem for the character of the complex Green's function b^n; cf. Section 1.1.3. Moreover, he also showed that

$$\left| \frac{b^n T_n \circ \mathfrak{z}}{\operatorname{Cap}(E)^n} - w_{n\mu} \right| \to 0 \tag{2.16}$$

uniformly on compact subsets of the universal cover. If E also contains Jordan arcs, he could only show that

$$1 \le \liminf_{n\to\infty} \frac{\|T_n\|_E}{\|w_{n\mu}\|_\infty \operatorname{Cap}(E)^n} \le \limsup_{n\to\infty} \frac{\|T_n\|_E}{\|w_{n\mu}\|_\infty \operatorname{Cap}(E)^n} \le 2$$

and had no results on the asymptotics of T_n. Under the restriction that E is a subset of the real line, i.e., a finite union of intervals, he could "sharpen" the lower bound and showed that

$$\lim_{n\to\infty} \frac{\|T_n\|_E}{\|w_{n\mu}\|_\infty \operatorname{Cap}(E)^n} = 2 \tag{2.17}$$

and recently, Christiansen, Simon and Zinchenko also proved (2.16) for this case. If (2.16) holds in a domain Ω, they introduced the notion that Ω has *Szegő-Widom asymptotics.*

Motivated by the real case, Widom conjectured that in fact the ratio in (2.17) should always be equal to 2 as soon as an arc component is present in E. In particular, if $\Omega = \overline{\mathbb{C}} \setminus E$ is simply connected this means that for any smooth arc E, one should have

$$\lim_{n\to\infty} \frac{\|T_n\|_E}{\operatorname{Cap}(E)^n} = 2.$$

S. Kalmykov and B. Nagy realized in 2013, see [64], that at that time (in 2013 not 1969) there existed already a counterexample to this conjecture. Indeed, Thiran and Detaille [62], considered in 1991 the Chebyshev polynomials associated to circular arcs

$$E_\alpha = \{\zeta \in \mathbb{T} : \ |\arg \zeta| \le \alpha\}, \quad 0 < \alpha < \pi$$

and showed that

$$\lim_{n\to\infty} \frac{\|T_n\|_{E_\alpha}}{\mathrm{Cap}(E_\alpha)^n} = 2\cos(\alpha/4)^2. \tag{2.18}$$

Note that one encounters a certain "continuity" of the right-hand side. For $0 < \alpha < \pi$, $2\cos(\alpha/4)^2$ varies between 2 and 1. That is, for small α the asymptotics behave like Widom proved for intervals. But if α is close to π, the value converges to 1, which corresponds to curves, hence in particular to $\mathbb{T}$. Further investigations have been made in order to understand the situations with arc components better. Totik [63] showed that whenever an arc component is present, then there exists a $\delta > 0$ such that

$$\liminf_{n\to\infty} \frac{\|T_n\|_E}{\mathrm{Cap}(E)^n} > 1 + \delta.$$

Totik and Yuditskii [64] in its turn showed that whenever there is at least one Jordan curve in E, then there exists $0 < \delta < 1$ such that

$$\limsup_{n\to\infty} \frac{\|T_n\|_E}{\mathrm{Cap}(E)^n \|w_{n\mu}\|_\infty} < 2 - \delta.$$

We study the Chebyshev polynomials associated to E_α more thoroughly. Our approach is completely different to the one in [62] and allows us to find:

(i) explicit asymptotics of the Chebyshev polynomials, T_n,

(ii) explicit asymptotics of the *upper envelope* of the family $\mathcal{P}_n(E_\alpha)$ of polynomials of degree at most n which are bounded by one in modulus on E_α; cf. (4.1).

Instead of the Chebyshev polynomial, it is again more convenient to consider the normalized polynomial $P_{n,\infty} = T_n/\|T_n\|_{E_\alpha}$. Set $\Omega_\alpha = \overline{\mathbb{C}} \setminus E_\alpha$ and let $B_{\Omega_\alpha}(\zeta, \zeta_0)$ be the complex Green's function with the standard normalization $B_{\Omega_\alpha}(\infty, \zeta_0) > 0$ if $\zeta_0 \neq \infty$ and $(\zeta B_{\Omega_\alpha}(\zeta, \infty))(\infty) > 0$ otherwise. Moreover, let us introduce the function

$$R_{E_\alpha}(\zeta) = (\zeta - e^{i\alpha})(\zeta - e^{-i\alpha}).$$

Theorem 2.13. *The domain Ω_α has Szegő-Widom asymptotics. That is, uniformly on compact subsets of Ω_α we have*

$$\lim_{n\to\infty} B_{\Omega_\alpha}(\zeta,\infty)^n P_{n,\infty}(\zeta) = \frac{1}{2}\left(1 - \frac{i}{e^{i\alpha}} \frac{\sqrt{R_{E_\alpha}(\zeta)}}{\zeta + 1}\right) \frac{B_{\Omega_\alpha}(\zeta,-1)}{B_{\Omega_\alpha}(\zeta,0)}. \tag{2.19}$$

The square-root of R_{E_α} is chosen such that the expression in the brackets has a zero at $\zeta = 0$. By $P_{n,\zeta_0} \in \mathcal{P}_n(E_\alpha)$ we denote those polynomials in $\mathcal{P}_n(E_\alpha)$ which have maximal value at the point ζ_0. Due to the symmetry of the domain, it suffices to consider the problem for $|\zeta_0| < 1$; see (4.6). By $\zeta^* = 1/\overline{\zeta}$ we denote the reflection with respect to the circle. The following theorem reveals the general structure of the solution:

Theorem 2.14. *Let $|\zeta_0| < 1$, K_{ζ_0} be the unique orthocircle to $\mathbb{T}$, which passes through ζ_0 and ζ_0^* and $\{u_0\} = K_{\zeta_0} \cap (\mathbb{T} \setminus E_\alpha)$. Then uniformly on compact subsets of Ω_α*

$$\lim_{n\to\infty} B_{\Omega_\alpha}(\zeta,\infty)^n P_{n,\zeta_0}(\zeta) = e^{i\phi}\frac{1}{2}\left(1 - c\frac{\sqrt{R_{E_\alpha}(\zeta)}}{\zeta - u_0}\right)\frac{B_{\Omega_\alpha}(\zeta,u_0)}{B_{\Omega_\alpha}(\zeta,\zeta_0^*)}, \tag{2.20}$$

where $c = \frac{\zeta_0^* - u_0}{\sqrt{R_E(\zeta_0^*)}}$, and ϕ is a proper phase.

We use the right half-plane $\mathbb{H}_+ := \{\lambda : \operatorname{Re}\lambda > 0\}$ as covering. In this case, we can give the covering map explicitly. Namely, let $\lambda : \Omega_\alpha \to \mathbb{H}_+ := \{\lambda : \operatorname{Re}\lambda > 0\}$ be defined by

$$\lambda(\zeta)^2 = \frac{\zeta e^{i\alpha} - 1}{\zeta - e^{i\alpha}}. \tag{2.21}$$

The normalizing factor $|B_{\Omega_\alpha}(\zeta_0,\infty)| = \exp\{-G_{\Omega_\alpha}(\zeta_0,\infty)\}$ is the ζ_0-depending analog of $\operatorname{Cap}(E_\alpha)$. By $L_n(\zeta)$ we denote the upper envelope of $\mathcal{P}_n(E_\alpha)$, i.e.,

$$L_n(\zeta) := \sup\{|P_n(\zeta)| : \ P_n \in \mathcal{P}_n(E_\alpha)\} = |P_{n,\zeta}(\zeta)|.$$

Theorem 2.15. *Let $\lambda = \lambda(\zeta)$ be defined as above, $\lambda_0 = \lambda(\zeta_0)$ and define the reproducing kernel $k_{\Omega_\alpha}(\zeta,\zeta_0)$ by*

$$k_{\Omega_\alpha}(\zeta,\zeta_0) = k_{Ch}(\lambda,\lambda_0) := \frac{2\sqrt{\lambda}\overline{\sqrt{\lambda_0}}}{(\sqrt{\lambda} + \overline{\sqrt{\lambda_0}})^2}.$$

Then

$$\lim_{n\to\infty} e^{-nG_{\Omega_\alpha}(\zeta,\infty)} L_n(\zeta) = k_{\Omega_\alpha}(\zeta,\zeta). \tag{2.22}$$

Remark 2.16. (i) The kernel $\frac{1}{(v+\overline{v}_0)^2}$ is up to normalization the Bergman kernel of the right half-plane $\mathbb{H}_+$. Hence, by setting $\lambda = v^2$, we see that k_{Ω_α} is indeed a reproducing kernel, since the product of reproducing kernels represents again a reproducing kernel.

(ii) This result gives a new interpretation of the value $2\cos(\alpha/4)^2$. In fact, it is the value of the same reproducing kernel k_{Ch}, for all α, on the covering, evaluated at different points, the image of infinity.

Finally, we would like to mention that the proofs will show that these results are universal in the following sense. In fact, one could instead of polynomials consider rational functions with a fixed collection of poles $C = \{c_1, \ldots, c_g\}$ outside of $\mathbb{D}$. The solution for the same problem for the class $\mathcal{F}_{n,\alpha}$ of rational functions with its only poles in C of order at most n is denoted by F_{n,ζ_0}. Let $B(\zeta) = \prod B_{\Omega_\alpha}(\zeta, c_k)$. Then for every choice of C we obtain the same result for the normalized limit as in (2.20), i.e.,

$$\lim_{n\to\infty} B(\zeta)^n F_{n,\zeta_0}(\zeta) = \frac{1}{2}\left(1 - c\frac{\sqrt{R_E(\zeta)}}{\zeta - u_0}\right)\frac{B_{\Omega_\alpha}(\zeta, u_0)}{B_{\Omega_\alpha}(\zeta, \zeta_0^*)},$$

where c and u_0 are defined as in Theorem 2.14. In particular, the upper envelope of this family, denoted by M_n, satisfies

$$\lim_{n\to\infty} e^{-n\sum G_{\Omega_\alpha}(\zeta, c_k)} M_n(\zeta) = k_{\Omega_\alpha}(\zeta, \zeta).$$

2.2.2 Polynomial Ahlfors problem

At the first glance, the phenomena encountered in Theorem 2.15 seems quite surprising. Why are in the setting of uniform approximation the asymptotics described by an object, which is related to a Hilbert space structure. On the other hand, comparable results are known in the setting of H^∞ functions. Namely:

1) Widom showed that if E is a finite union of Jordan regions, the asymptotics can be given by a related character automorphic H^∞ problem; cf. Theorem 1.17 and Theorem 1.18. Due to generalizations of results of Abrahamse [1], it is known that this extremal character automorphic functions in its turn can be expressed in terms of reproducing kernels; cf. Proposition 4.16.

The other result is related to what we call *Ahlfors problem.*

Problem 3. *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain with boundary $E = \partial\Omega$ and $z_0 \in \Omega$. Find:*

$$\mathcal{A}_\Omega(z_0) := \sup\{|\Psi'(z_0)| : \ \Psi \in H^\infty(\Omega), \Psi : \Omega \to \mathbb{D}, \Psi(z_0) = 0\}.$$

It is known that, up to a unimodular factor, there exists a unique maximizer, Ψ_{z_0}; see e.g. [57, Section 8.8.].

In 1947, Ahlfors [2] first considered this kind of problem for n-connected domains and proved existence and uniqueness of Ψ_{z_0}. For this reason Ψ_{z_0} is called the *Ahlfors function* of the domain Ω and the point z_0. Note that we already introduced the Ahlfors function for infinity, $\Psi = \Psi_\infty$, in Section 2.1.

2) Garabedian 1949 [27], in its turn showed that if Ω is bounded and the boundary consists of finitely many analytic Jordan curves, then $\mathcal{A}_\Omega(z_0)$ is the diagonal of the Szegő kernel of Ω, i.e., the reproducing kernel of $H^2(\Omega)$.

This is, indeed, a generalization of the classical Schwarz lemma: if Ω is the right half-plane, $\mathbb{H}_+$, then

$$\mathcal{A}_\Omega(\lambda_0) = \left(\frac{\lambda - \lambda_0}{\lambda + \bar{\lambda}_0}\right)'_{\lambda=\lambda_0} = \frac{1}{\lambda_0 + \bar{\lambda}_0} = k_{Sz}(\lambda_0, \lambda_0; \Omega).$$

In addition to the affirmative example of E_α from the previous section, these two observations motivated us to make the following conjecture:

Conjecture 1. *Asymptotics for Chebyshev polynomials in a complex domain Ω can be given in terms of the reproducing kernels of a suitable Hilbert space of analytic functions in Ω.*

Having in mind the conformal invariance of our conjecture, we will study simultaneously the following three problems, naturally related to the Ahlfors problem.

Problem 4. *Let E_J be a real compact consisting of $g+1$ non-degenerated intervals, $E_J = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{i=1}^{g}(\mathbf{a}_i, \mathbf{b}_i)$. Let $\mathcal{P}_n(E_J)$ be the collection of polynomials of degree n bounded in absolute value by 1 on E_J. Define*

$$A_n(z; E_J) = \sup\{|P'(z)| : \; P \in \mathcal{P}_n(E_J), \; P(z) = 0\}, \quad z \in \mathbb{C} \setminus E_J.$$

Find asymptotics for $A_n(z; E_J)$ as $n \to \infty$.

Problem 5. *Let E_T be a system of arcs, $E_T = \mathbb{T} \setminus \{e^{iz} : z \in \cup_{i=1}^{g}(\mathbf{a}_i, \mathbf{b}_i)\}$. Let $\mathcal{P}_n(E_T)$ be a collection of polynomials of degree n bounded in absolute value by 1 on E_T. Define*

$$A_n(\zeta; E_T) = \sup\{|P'(\zeta)| : \; P \in \mathcal{P}_n(E_T), \; P(\zeta) = 0\}, \quad \zeta = e^{iz} \in \mathbb{C} \setminus E_T.$$

Find asymptotics for $A_n(\zeta; E_T)$ as $n \to \infty$.

Problem 6. *Let $E_S = \mathbb{R}_+ \setminus \cup_{i=1}^{g}(\mathbf{a}_i, \mathbf{b}_i)$. Let $\mathcal{E}_\ell(E_S)$ be the collection of entire functions $F(z)$ of order $1/2$, of exponential type at most ℓ and bounded in absolute value by 1 on E_S, that is,*

$$|F(z)| \le C(\ell')e^{\ell'\sqrt{|z|}}, \; \forall \; \ell' > \ell, \quad |F(z)| \le 1 \textit{ for } z \in E_S.$$

Define

$$A_\ell(z; E_S) = \sup\{|F'(z)| : \; F \in \mathcal{E}_\ell(E_S), \; F(z) = 0\}, \quad z \in \mathbb{C} \setminus E_S.$$

Find asymptotics for $A_\ell(z; E_S)$ as $\ell \to \infty$.

Like for Chebyshev polynomials, cf. Theorem 1.18, we first studied the corresponding character-automorphic problem for uniformly bounded analytic functions. For all three cases, i.e., $\Omega_{J,T,S} = \overline{\mathbb{C}} \setminus E_J, \overline{\mathbb{C}} \setminus E_T, \mathbb{C} \setminus E_S$ we can define $H^\infty_\Omega(\beta)$ as in Section 1.1.3.

Problem 7. *Let $\Omega = \overline{\mathbb{C}} \setminus E_J, \overline{\mathbb{C}} \setminus E_T, \mathbb{C} \setminus E_S$, $z_0 \in \Omega$ and $\beta \in \pi(\Omega)^*$. Find*

$$\mathcal{A}_\Omega(z_0, \beta) := \sup\{|\Psi'(z_0)| : \; \Psi \in H^\infty_\Omega(\beta), \|\Psi\|_E \le 1, \Psi(z_0) = 0\}.$$

To study this problem, it is more suitable to use $\mathbb{C}_+$ as covering domain. Hence, let $\Gamma = \Gamma_{J,T,S}$ be the Fuchsian groups such that $\Omega_{J,T,S} \simeq \mathbb{C}_+/\Gamma_{J,S,T}$. The action of the group Γ will be discussed in more detail in Section 4.2.1.

In the following, we will present a solution of Problem 7 for $\Omega = \Omega_S$. Note that due to the conformal invariance of the setting, this provides a solution for all three cases. Since infinity is not an internal point of Ω, the Green's function should be substituted by its analog for boundary points, which is the so-called symmetric

Martin's function, $M_\Omega(z)$. It is a positive, symmetric (i.e., $M_\Omega(\overline{z}) = M_\Omega(z)$) harmonic function in Ω, which vanishes on E. For our domain it is unique up to a positive multiplier. References for Martin's function are e.g. [13, 25, 31]. We will give a definition of the Martin's function in terms of conformal mappings on comb domains in Section 4.2.1, where we chose the normalization $M_\Omega(x) \sim \sqrt{-x}$ as $x \to -\infty$. By $\mathcal{M} \circ \lambda = i(M_\Omega + i * M_\Omega)$, it induces an analytic function on Ω, respectively on the universal cover. Note that in contrary to Green's function already $\mathcal{M}$ is analytic in Ω, since infinity is not an internal point. The complex Martin's function, defined by $\exp(i\mathcal{M}(\lambda))$, is character automorphic and we will denote its character by μ_S.

Let $\{\mathbf{d}_i\}$ be the critical points of M_Ω. Again as for Green's function there is exactly one critical point in each gap $(\mathbf{a}_i, \mathbf{b}_i)$. Due to Pommerenke, $\mathcal{M}'$ is of bounded characteristic with no singular inner part in Ω and therefore admits a factorization into a Blaschke product, $\mathcal{W}$, and outer part, $\mathcal{M}_o$. Let $\varpi(\lambda) = \sqrt{\mathcal{M}_o(\lambda)}$. Hence, by definition

$$\mathcal{M}'(\lambda) = \mathcal{W}(\lambda)\varpi(\lambda)^2,$$

where $\mathcal{W}$ is given by

$$\mathcal{W}(\lambda) = \prod_{i=1}^{g} b_{\mathbf{d}_i}(\lambda).$$

Let $\alpha_{\mathcal{W}} = \sum_{i=1}^{g} \mu_{\mathbf{d}_i}$ denote its character. All this is collected in more detail in Theorem 4.12.

Let $H^p(\mathbb{C}_+)$ be the standard Hardy space in $\mathbb{C}_+$,

$$\|f\|_p^p = \frac{1}{2\pi}\int_{\mathbb{R}} |f(\xi)|^p \mathrm{d}\xi, \quad f \in H^p(\mathbb{C}_+),$$

with a proper modification for $p = \infty$. For a fixed character $\alpha \in \Gamma^*$ we introduce

$$H^p(\alpha) = \{f \in H^p(\mathbb{C}_+) : \quad f(\gamma(\lambda)) = \alpha(\gamma)f(\lambda), \gamma \in \Gamma\}.$$

Note that $H^2(\alpha)$ does not coincide with our "standard" definition of H^2_Ω, because, roughly speaking, $L^2(\mathrm{d}x)$ and $L^2(\omega(i, \mathrm{d}x; \mathbb{C}_+))$ are not the same. Moreover, for this section instead of additive, we will use multiplicative notation for the characters. The point evaluation functional is bounded in H^2. By $k^\alpha_{\lambda_0}(\lambda) = k^\alpha(\lambda, \lambda_0)$ we denote the reproducing kernel in $H^2(\alpha)$, $\langle f, k^\alpha_{\lambda_0}\rangle = f(\lambda_0)$ for all $f \in H^2(\alpha)$.

Definition 2.17. We define the Szegő kernel (corresponding to the given group Γ and its character α), by

$$k_{Sz}(\lambda, \lambda_0; \Gamma, \alpha) = k^\alpha_{Sz}(\lambda, \lambda_0) = k^{\alpha\nu^{-1}}(\lambda, \lambda_0)\varpi(\lambda)\overline{\varpi(\lambda_0)},$$

where ν denotes a certain fixed root of $\alpha_{\mathcal{W}}^{-1}$, $\nu^2 = \alpha_{\mathcal{W}}^{-1}$.

It is the reproducing kernel for the Hilbert space of forms $A^2_1(\Gamma, \alpha)$, cf. Definition 4.14.

Theorem 2.18. *Let*

$$\mathcal{A}(\lambda_0,\beta) := \sup\{|w'(\lambda_0)| : \ w \in H^\infty(\beta), \|w\|_E \le 1, w(\lambda_0)=0\}.$$

Then the solution is given in terms of the Szegő kernels

$$\mathcal{A}(\lambda_0,\beta) = \min_{\alpha^2=\beta} k^{\alpha}_{Sz}(\lambda_0,\lambda_0) = k^{\alpha(\lambda_0)}_{Sz}(\lambda_0,\lambda_0). \tag{2.23}$$

If $\beta = 1_{\Gamma^*}$ the above minimum is assumed on the same half period $\mathfrak{j} \in \Gamma^*$ for all $\lambda_0 \in \mathbb{C}_+$, that is, see (4.21), the Garabedian formula holds

$$\mathcal{A}(\lambda_0, 1_{\Gamma^*}) = \mathcal{A}(\lambda_0, 1_{\Gamma^*_{\mathfrak{j}}}) = k_{Sz}(\lambda_0,\lambda_0;\Gamma_{\mathfrak{j}}, 1_{\Gamma^*_{\mathfrak{j}}}). \tag{2.24}$$

Generally, the extremal character $\alpha(\lambda_0)$, $\alpha(\lambda_0)^2=\beta$, depends on λ_0.

Note that

$$\mathcal{A}(\lambda_0,\beta) = \mathcal{A}_\Omega(z(\lambda_0),\beta)\left|\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right|.$$

If z_0 is in the "gaps" of the corresponding domains, we proved an analog of Theorem 1.18.

Theorem 2.19. *Let $z_0 \in \mathbb{R}\setminus E_{\mathbb{R}}$, $\zeta_0 \in \mathbb{T}\setminus E_T$ or $z_0 \in \mathbb{R}\setminus E_S$. Then the solutions of Problems 4, 5, 6 are given by*

$$\lim_{n\to\infty}\left\{e^{-nG_{\Omega_J}(z_0,\infty)}A_n(z_0,E_J) - \frac{1}{2}\mathcal{A}_{\Omega_J}(z_0, n\mu_J)\right\} = 0, \tag{2.25}$$

$$\lim_{n\to\infty}\left\{e^{-nG_{\Omega_T}(\zeta_0,\infty)}A_n(\zeta_0,E_T) - \frac{1}{2}\mathcal{A}_{\Omega_T}(\zeta_0, n\mu_T)\right\} = 0, \tag{2.26}$$

$$\lim_{\ell\to\infty}\left\{e^{-\ell M_{\Omega_S}(z_0)}A_\ell(z_0,E_S) - \frac{1}{2}\mathcal{A}_{\Omega_S}(z_0, \ell\mu_S)\right\} = 0, \tag{2.27}$$

where $G_\Omega(z,\infty)$, $G_\Omega(\zeta,\infty)$ and $M_\Omega(z)$ are the Green's and Martin's functions in the corresponding domains and μ_J,μ_T,μ_S the corresponding characters.

For multiply connected domains, we solved Problems 4 and 5 in vicinities of the "gaps". In order to avoid too many definitions, we will present the general result only for the set $E = E_J$. Let $\Omega = \overline{\mathbb{C}}\setminus E$ and

$$E_k = E \cap [\mathbf{b}_0, \mathbf{a}_k], \quad 1 \le k \le g.$$

Let γ_k be a set of generators starting at infinity and encircling the sets E_k. Finally, for a set $\mathfrak{X} = \{\mathbf{x}_i\}_{i=0}^g$, where $\mathbf{x}_i \in (\mathbf{a}_i, \mathbf{b}_i)$, recall the special functions

$$U_{\mathfrak{X}}(z) = \prod_{i=0}^g (z-\mathbf{x}_i), \quad \text{and} \quad R_E(z) = \prod_{i=0}^g (z-\mathbf{a}_i)(z-\mathbf{b}_i).$$

Theorem 2.20. *Let μ_J be the character of the complex Green's function $B_\Omega(z,\infty) = \exp(-(G_\Omega(z,\infty) + i * G_\Omega(z,\infty)))$. Assume that μ_J is in a generic position, that is, $\operatorname{clos}\{\mu_J^n\}_{n\in\mathbb{Z}} = \pi(\Omega)^*$. Then there exists an open set $\mathcal{V} = \mathcal{V}_1\times\mathcal{V}_2 \subset \pi(\Omega)^*\times\mathbb{C}_+$ such that*

$$\lim_{k\to\infty} e^{-n_k G_\Omega(z_0,\infty)} A_{J,n_k}(z_0) = Y(z_0,\beta) := \frac{1}{2\operatorname{Im} z_0} e^{-\sum_{j=0}^g G_\Omega(\mathbf{x}_j, z_0)}, \tag{2.28}$$

where $\{\beta, z_0\} \in \mathcal{V}$, $\beta = \lim_{k\to\infty} \mu_J^{n_k}$, *is related to* $\{\mathbf{x}_0, \dots, \mathbf{x}_g, \rho\}$ *by*

$$\sum_{j=0}^{g} \omega(\mathbf{x}_j, E_k; \Omega) = \beta(\gamma_k), \quad \text{for } 1 \le k \le g \tag{2.29}$$

and

$$-i\rho = \frac{U_{\mathfrak{X}}(z_0)}{\sqrt{R_E(z_0)}}. \tag{2.30}$$

Compare the similarity between the expression in brackets in (2.20) and (2.30). This theorem needs certain explanations, which we will provide in the following remark.

Remark 2.21. (i) One should read this theorem as a local statement. We fix a special gap, say let us fix $\mathbf{x}_0 > \mathbf{a}_0$. Then we choose an open set $\mathcal{V}_1$ such that for all $\beta \in \mathcal{V}_1$ we have for the solution of (2.29) that $\mathbf{x}_j \in (\mathbf{a}_j, \mathbf{b}_j)$. Then we can even choose an open interval containing $\mathbf{x}_0$ such that this remains valid ($\mathbf{x}_j$ depends continuously on $\mathbf{x}_0$). Finally, for sufficiently small ρ, (2.30) defines z_0 in a vicinity of $\mathbf{x}_0$.

(ii) We will see in Section 4.2.5 that (2.29), (2.30) should be in fact understood as an extended Abel-Jacobi inversion problem. Compare also (1.31).

For simply connected domains, it turns out that Theorem 2.20 provides a global solution and indeed also leads to an affirmative answer to our conjecture. In fact, the asymptotics of Problem 4 and Problem 5 can be given by the same reproducing kernel on the universal covering. Let

$$k_{Sz}(\lambda, \lambda_0) = k_{Sz}(\lambda, \lambda_0; \mathbb{H}_+) := \frac{1}{\lambda + \overline{\lambda}_0}$$

be the Szegő kernel of the right half-plane and

$$k_{Ch}(\lambda, \lambda_0) = \frac{2\sqrt{\lambda}\overline{\sqrt{\lambda_0}}}{(\sqrt{\lambda} + \overline{\sqrt{\lambda_0}})^2}$$

the reproducing kernel from the Chebyshev type asymptotics in Theorem 2.15. Moreover, let $E_J = [-2, 2]$ and $E_{\mathbb{T}} = E_\alpha = \{\zeta \in \mathbb{T} : |\arg \zeta| \le \alpha\}$ and correspondingly the domains $\Omega_{J,T}$. Define the conformal mappings $\lambda_J : \Omega_J \to \mathbb{H}_+$, $\lambda_T : \Omega_T \to \mathbb{H}_+$ by

$$z = 2\frac{\lambda_J^2 + 1}{\lambda_J^2 - 1}, \quad \lambda_T^2 = \frac{\zeta e^{i\alpha} - 1}{\zeta - e^{i\alpha}}.$$

Theorem 2.22. *With the notation from above, we have the following uniform description of the asymptotics:*

(i) $$\lim_{n\to\infty} e^{-nG_{\Omega_J}(z,\infty)} A_n(z; [-2, 2]) \left|\frac{\mathrm{d}z}{\mathrm{d}\lambda_J}\right| = k_{Sz}(\lambda_J, \lambda_J) k_{Ch}(\lambda_J, \lambda_J),$$

(ii) $$\lim_{n\to\infty} e^{-nG_{\Omega_T}(z,\infty)} A_n(\zeta; E_\alpha) \left|\frac{\mathrm{d}\zeta}{\mathrm{d}\lambda_T}\right| = k_{Sz}(\lambda_T, \lambda_T) k_{Ch}(\lambda_T, \lambda_T).$$

Remark 2.23. (i) We would like to point out that in contrary to the kernel k_{Ch}, which is related to maximizing the point evaluation, for this kernel $\mathbb{H}_+$ is indeed the maximal domain on which it is analytic. This is understandable from the following point of view. Given a reproducing kernel Hilbert space, generically $\|k_{z_0}\|$ is not uniformly bounded as a function of z_0. But since for the polynomial upper envelope L_n, we have

$$|B_{\Omega_\alpha}(\zeta, \infty)^n P(\zeta)| \leq 1$$

for all $\zeta \in \Omega_\alpha$ and all $P \in \mathcal{P}(E_\alpha)$ the limit is forced to be bounded by one. This is not the case if one maximizes the value of $P'(\zeta_0)$.

Chapter 3

Periodic GMP matrices

3.1 Direct spectral theory of periodic GMP matrices

In this section we present the proofs of Section 2.1. All definitions and notations which are not repeated at this place can also be found in this section.

Let A be a one-block periodic GMP matrix. Due to [72, Proposition 5.5], $\{e_{-1}, \tilde{e}_0\}$ span a cyclic subspace for A. Like in (1.22), we can represent A as

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} + a_0(\langle \cdot, e_{-1}\rangle \tilde{e}_0 + \langle \cdot, \tilde{e}_0\rangle e_{-1}),$$

where $a_0 = \|\vec{p}\|$. Hence, defining

$$R(z) = \begin{bmatrix} \langle (A-z)^{-1}e_{-1}, e_{-1}\rangle & \langle (A-z)^{-1}\tilde{e}_0, e_{-1}\rangle \\ \langle (A-z)^{-1}e_{-1}, \tilde{e}_0\rangle & \langle (A-z)^{-1}\tilde{e}_0, \tilde{e}_0\rangle \end{bmatrix},$$

we obtain

$$R(z) = \begin{bmatrix} r_-(z)^{-1} & a_0 \\ a_0 & r_+(z)^{-1} \end{bmatrix}^{-1}, \tag{3.1}$$

where

$$r_-(z) = \langle (A_- - z)^{-1}e_{-1}, e_{-1}\rangle, \qquad r_+(z) = \langle (A_+ - z)^{-1}\tilde{e}_0, \tilde{e}_0\rangle.$$

The following theorem is an analogue of (1.26) for periodic GMP matrices.

First, we introduce some notations, which will be used in the proof. For $\vec{x} \in \mathbb{R}^{g+1}$, we define

$$s_k\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{g-k} \end{bmatrix}.$$

Moreover, let the M_k's be upper triangular matrices such that

$$B(\vec{p},\vec{q}) - \vec{p}(\vec{q})^* = M(\vec{p},\vec{q}) := M_0 = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} + (-\vec{p}q_g + \vec{q}p_g)\delta_g^*$$

and

$$M_k = \begin{bmatrix} M_{k+1} & 0 \\ 0 & \mathbf{c}_{g+1-k} \end{bmatrix} + (-s_k \vec{p} q_{g-k} + s_k \vec{q} p_{g-k})(s_k \delta_{g-k})^* \quad \text{for} \quad 1 \le k \le g-1.$$

Theorem 3.1. *Let*

$$\begin{bmatrix} R(z,p,p) & R(z,g,p) \\ R(z,p,g) & R(z,g,g) \end{bmatrix} = \begin{bmatrix} \langle (B(\vec{p},\vec{q})-z)^{-1}\vec{p},\vec{p}\rangle & \langle (B(\vec{p},\vec{q})-z)^{-1}\vec{\delta}_g,\vec{p}\rangle \\ \langle (B(\vec{p},\vec{q})-z)^{-1}\vec{p},\vec{\delta}_g\rangle & \langle (B(\vec{p},\vec{q})-z)^{-1}\vec{\delta}_g,\vec{\delta}_g\rangle \end{bmatrix}.$$

Let A be a periodic GMP matrix, $r_+(z)$ the resolvent function of A_+, i.e.,

$$r_+(z) = \left\langle (A_+ - z)^{-1}\tilde{e}_0, \tilde{e}_0 \right\rangle.$$

Then we have

$$a_0^2 r_+(z) = \frac{a_0^2 r_+(z)\tilde{\mathfrak{A}}_{11}(z) + \tilde{\mathfrak{A}}_{12}(z)}{a_0^2 r_+(z)\tilde{\mathfrak{A}}_{21}(z) + \tilde{\mathfrak{A}}_{22}(z)}, \tag{3.2}$$

where $a_0^2 = \|\vec{p}\|^2$ and

$$\tilde{\mathfrak{A}}(z) = \begin{bmatrix} \tilde{\mathfrak{A}}_{11}(z) & \tilde{\mathfrak{A}}_{12}(z) \\ \tilde{\mathfrak{A}}_{21}(z) & \tilde{\mathfrak{A}}_{22}(z) \end{bmatrix} = \frac{1}{R(z,p,g)} \begin{bmatrix} R(z,p,p)R(z,g,g) - R(z,p,g)^2 & -R(z,p,p) \\ R(z,g,g) & -1 \end{bmatrix}.$$

Proof. We write

$$A_+ = \begin{bmatrix} B(\vec{p},\vec{q}) & 0 \\ 0 & A_+ \end{bmatrix} + a_0(\langle \cdot, e_g\rangle \tilde{e}_1 + \langle \cdot, \tilde{e}_1\rangle e_g),$$

where $\tilde{e}_1 = S^{g+1}\tilde{e}_0$ and apply the Sherman–Morrison–Woodbury formula (cf. [29, Section 2.1.3]) to prove the theorem. □

Theorem 3.2. *Let $\tilde{\mathfrak{A}}$ be defined as in Theorem 3.1 and $\mathfrak{A}$ be the transfer matrix of A. Then we have $\tilde{\mathfrak{A}}(z) = \mathfrak{A}(z)$.*

Proof. First, we represent $B(\vec{p},\vec{q})$ as a one-dimensional perturbation of a lower diagonal matrix. Applying the Sherman–Morrison–Woodbury formula again, leads to a representation of $\tilde{\mathfrak{A}}$ in terms of M_0. Using that M_0 is a lower triangular matrix, we obtain

$$\tilde{\mathfrak{A}}(z) = \mathfrak{A}_0(z)\mathfrak{a}(z; p_g, q_g),$$

where

$$\mathfrak{A}_0(z) = I - \begin{bmatrix} \langle (M_1 - z)^{-1}u_1\vec{p}, u_1\vec{p}\rangle & \langle (M_1 - z)^{-1}u_1\vec{q}, u_1\vec{p}\rangle \\ \langle (M_1 - z)^{-1}u_1\vec{p}, u_1\vec{q}\rangle & \langle (M_1 - z)^{-1}u_1\vec{q}, u_1\vec{q}\rangle \end{bmatrix} \mathrm{j}.$$

Using again that all M_j's are lower triangular matrices, we find that

$$\mathfrak{A}_{j-1}(z) = \mathfrak{A}_j(z)\mathfrak{a}(z, \mathbf{c}_{g+1-j}; p_{g-j}, q_{g-j}),$$

where

$$\mathfrak{A}_{j-1}(z) = I - \begin{bmatrix} \langle (M_j - z)^{-1}u_j\vec{p}, u_j\vec{q}\rangle & \langle (M_j - z)^{-1}u_j\vec{q}, u_j\vec{p}\rangle \\ \langle (M_j - z)^{-1}u_j\vec{p}, u_j\vec{q}\rangle & \langle (M_j - z)^{-1}u_j\vec{q}, u_j\vec{q}\rangle \end{bmatrix} \mathrm{j}.$$

□

Remark 3.3.

1. Note that $\mathfrak{A}$ is normalized such that $\det \mathfrak{A} = 1$.
2. For a general (non periodic) GMP matrix the only difference in (3.2) is that in the right-hand side a_0 is replaced by $\|\vec{p}_1\|$ and r_+ by $r_+^{(1)}$, which would be the correct analog of (1.26). Therefore, we still call $\mathfrak{A}$ transfer matrix for the given GMP matrix.
3. Unlike Jacobi matrices, we consider the relation between the resolvent function of the initial and the $g+1$-shifted GMP matrix, since this shift preserves its structure.

Applying the same calculations to r_- leads to the following theorem.

Theorem 3.4. *For a periodic GMP matrix A, let*

$$\mathfrak{A}_-(z) = \begin{bmatrix} \mathfrak{A}_{11}^-(z) & \mathfrak{A}_{12}^-(z) \\ \mathfrak{A}_{21}^-(z) & \mathfrak{A}_{22}^-(z) \end{bmatrix}$$

be the transfer matrix of r_-, i.e.,

$$r_-(z) = \frac{r_-(z)\mathfrak{A}_{11}^-(z) + \mathfrak{A}_{12}^-(z)}{r_-(z)\mathfrak{A}_{21}^-(z) + \mathfrak{A}_{22}^-(z)}. \tag{3.3}$$

Then it is of the form

$$\mathfrak{A}_-(z) = \mathfrak{a}_-(z, \infty; p_g, q_g)\mathfrak{a}_-(z, c_g; p_{g-1}, q_{g-1}) \cdots \mathfrak{a}_-(z, c_1; p_0, q_0),$$

where

$$\mathfrak{a}_-(z, \infty; p, q) = \begin{bmatrix} 0 & -\frac{1}{p} \\ p & \frac{z-pq}{p} \end{bmatrix} \qquad \textit{and} \qquad \mathfrak{a}_-(z, \mathbf{c}; p, q) = I - \frac{1}{\mathbf{c} - z} \begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} q & p \end{bmatrix} \mathfrak{j}.$$

Proof. The proof is the same as for r_+, but in order to extract $\mathfrak{a}_-(z, \infty; p, q)$ in the first step, one has to write the "mirrored" B-block as a one-dimensional perturbation of an upper triangular matrix. □

The following corollary will be an important ingredient in the proof of Theorem 2.10.

Corollary 3.5. *With the notation from above the entries of $\mathfrak{A}$ and $\mathfrak{A}_-$ are related by*

$$\mathfrak{A}_{11}(z) = \mathfrak{A}_{11}^-(z), \qquad \mathfrak{A}_{12}(z) = -\mathfrak{A}_{21}^-(z), \qquad \mathfrak{A}_{21}(z) = -\mathfrak{A}_{12}^-(z), \qquad \mathfrak{A}_{22}(z) = \mathfrak{A}_{22}^-(z).$$

Proof. This follows by the relation

$$\mathfrak{A}(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathfrak{A}_-(\overline{z})^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

□

Clearly, (3.2) is a quadratic equation for $a_0^2 r_+$. Let $\Delta^A(z) = \operatorname{tr}\mathfrak{A}(z)$ and $V(z) = \mathfrak{A}_{11}(z) - \mathfrak{A}_{22}(z)$. Using the normalization of $\mathfrak{A}$, we see that for all $z \in \mathbb{C}_+$

$$a_0^2 r_+(z) = \frac{1}{2\mathfrak{A}_{21}(z)}\left(V(z) + \sqrt{\Delta^A(z)^2 - 4}\right), \tag{3.4}$$

where one takes the branch of the square root with $\sqrt{\Delta^A(z)^2 - 4} = \Delta^A(z) + \mathcal{O}\left(\frac{1}{\Delta^A(z)}\right)$ near $z = \infty$.

Lemma 3.6. *The function r_-^{-1} is the second solution of* (3.2)*. That is,*

$$\frac{1}{r_-(z)} = \frac{1}{2\mathfrak{A}_{21}(z)}\left(V(z) - \sqrt{\Delta^A(z)^2 - 4}\right),$$

Proof. Due to (3.3), we have

$$r_-(z) = \frac{\mathfrak{A}_{22}^- r_- - \mathfrak{A}_{12}^-}{-\mathfrak{A}_{21}^- r_- + \mathfrak{A}_{11}^-}(z).$$

Using Corollary 3.5, we see that r_-^{-1} is a solution of (3.2). That r_-^{-1} is distinct form $a_0^2 r_+$ on $\mathbb{C}_+$ follows since $\operatorname{Im} r_-^{-1} < 0$, while $\operatorname{Im} a_0^2 r_+ > 0$ on $\mathbb{C}_+$. □

Lemma 3.7. *The resolvent function of a periodic* GMP *matrix A admits the following representation:*

$$R(z) = \frac{1}{2a_0^2\sqrt{\Delta^A(z)^2 - 4}}\begin{bmatrix} -2a_0^2\mathfrak{A}_{21}(z) & a_0 V(z) \\ a_0 V(z) & 2\mathfrak{A}_{12}(z)\end{bmatrix} + \frac{1}{2a_0}\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}.$$

Proof. This is a consequence of (3.1) and Lemma 3.6. □

Proof of Theorem 2.10. We have $\sigma_{ac}(A) = \{z \in \mathbb{C} : \Delta^A(z) \in [-2, 2]\}$. Pure point spectrum can only appear at poles of $\mathfrak{A}_{ij}$. But since $-\operatorname{Res}_{\mathbf{c}_k} \Delta^A(z) > 0$, this is not possible. □

3.2 Functional models for periodic GMP matrices

In the construction of the functional model, we use the notation of the preliminary chapter. Recall in particular that b, μ, k^α denote the Blaschke factor, character and reproducing kernel related to infinity.

We have already mentioned in the introduction that the Ahlfors function, Ψ, will serve as an analogue of Φ for general finite gap sets. Due to Lemma 2.2 and its proof, we obtain the following properties of Ψ:

(i) $|\Psi| < 1$ in $\overline{\mathbb{C}} \setminus E$ and $|\Psi| = 1$ on E.

(ii) $\Psi(z) = 0 \Leftrightarrow z \in \{\mathbf{c}_1, \ldots, \mathbf{c}_g\} \cup \{\infty\}$.

All this implies that

$$\log\frac{1}{|\Psi(z)|}=G_\Omega(z)+\sum_{j=1}^{g}G_\Omega(z,\mathbf{c}_j).$$

Therefore, $\Psi(\mathfrak{z}(\zeta))=b(\zeta)\prod_{j=1}^{g}b_{\mathbf{c}_j}(\zeta)$. In particular, $\mu+\sum_{j=1}^{g}\mu_{\mathbf{c}_j}=\mathbf{0}_{\Gamma^*}$. We define the permutation π by demanding that $\mathbf{c}_j\in(\mathbf{a}_{\pi(j)},\mathbf{b}_{\pi(j)})$. Moreover, we fix generators of the group Γ, $\{\gamma_j\}_{j=1}^{g}$, where γ_j corresponds to a closed curve, which starts at ∞ and passes through the gap $(\mathbf{a}_{\pi(j)},\mathbf{b}_{\pi(j)})$. Due to the symmetry of $\mathbb{C}_+$ and $\mathbb{C}_-$ we can choose a fundamental domain, $\mathcal{F}\subset\mathbb{D}$, of the group, which is symmetric w.r.t. $\zeta\mapsto\overline{\zeta}$. Choosing $\zeta_j\in(\mathcal{F}\cap\mathfrak{z}^{-1}(\mathbf{c}_j))$, we have $\overline{\zeta}_j=\gamma_j(\zeta_j)$. Let us fix these $\zeta_j\in\mathbb{D}$.

The relation (2.5) together with the factorization of Ψ into Blaschke products suggests to consider the following counterpart of (1.34). Let $\beta_n=\alpha-\sum_{k=1}^{n}\mu_{\mathbf{c}_k}$ and $\eta_n=\sqrt{\exp(-2\pi i\beta_n(\gamma_{n+1}))}$. Then

$$H^2(\alpha)=\{k^\alpha_{\zeta_1},\dots,k^\alpha_{\zeta_g},k^\alpha\}\oplus\Psi H^2(\alpha)=\{f_0^\alpha\}\oplus\dots\oplus\{f_g^\alpha\}\oplus\Psi H^2(\alpha),$$

where

$$f_0^\alpha=\eta_0\frac{k^\alpha_{\zeta_1}}{\sqrt{k^\alpha_{\zeta_1}(\zeta_1)}},\qquad f_1^\alpha=\eta_1\frac{b_{\mathbf{c}_1}k^{\alpha-\mu_{\mathbf{c}_1}}_{\zeta_2}}{\sqrt{k^{\alpha-\mu_{\mathbf{c}_1}}_{\zeta_2}(\zeta_2)}},\qquad\dots,\qquad f_g^\alpha=\frac{\prod_{j=1}^{g}b_{\mathbf{c}_j}k^{\alpha+\mu}}{\sqrt{k^{\alpha+\mu}(0)}}.$$

Remark 3.8. The unimodular factors η_n are chosen such that the matrix representation of multiplication by $\mathfrak{z}$ w.r.t. this basis, i.e., the corresponding GMP matrix is real. Note that the square root of $e^{-2\pi i\beta_n(\gamma_{n+1})}$ is defined up to the choice of a multiplicative constant ±1. Thus, in fact to a given character α we associate 2^g bases and therefore GMP matrices. This is the reason for the identification $(p_j,q_j)\to(-p_j,-q_j)$ in Theorem 2.5.

Theorem 3.9. *The system of functions*

$$f_n^\alpha=f_n^\alpha(\zeta;\mathbf{c}_1,\dots,\mathbf{c}_g)=\Psi^m f_j^\alpha,\qquad n=(g+1)m+j,\qquad 0\le j\le g,$$

(i) *forms an orthonormal basis in $H^2(\alpha)$ for $n\in\mathbb{Z}_{\ge0}$ and*

(ii) *forms an orthonormal basis in $L^2(\alpha)$ for $n\in\mathbb{Z}$.*

Proof. By construction, this system is orthogonal. The completeness of the second system follows by Lemma 1.36. □

Proof of Theorem 2.5. We show that $A(\alpha)$ is a GMP matrix. Clearly, $A(\alpha)$ is $g+1$-periodic. Let P_-^α be the projection onto $L^2(\alpha)\ominus H^2(\alpha)$. Then we have

$$P_-^\alpha(\mathfrak{z}f_n^\alpha)=p_n(\alpha)f_{-1}^\alpha,\qquad j=0,\dots,g.$$

Since $\mathfrak{z}$ is self-adjoint and $A(\alpha)$ has constant block-coefficients, this shows that $A(\vec{p})$ has the right structure.We have

$$p_n(\alpha)=\langle\mathfrak{z}f_n^\alpha,f_{-1}^\alpha\rangle=\frac{(\mathfrak{z}b)(0)\eta_n\prod_{k=1}^{n}b_{\mathbf{c}_k}(0)k^{\beta_n}(0,\zeta_{n+1})}{\sqrt{k^{\beta_n}_{\zeta_{n+1}}(\zeta_{n+1})k^{\alpha+\mu}(0)}}$$

for $0 \le n \le g$. Since $k^\alpha(\bar{\zeta_n}) = \overline{k^\alpha(\zeta_n, 0)}$, we obtain

$$\overline{k^{\beta_n}(0,\zeta_{n+1})} = k^{\beta_n}(\zeta_{n+1},0) = \overline{k^{\beta_n}(\gamma_{n+1}(\zeta_{n+1}),0)} = \eta_n^2 k^{\beta_n}(0,\zeta_{n+1}).$$

Thus, $p_n(\alpha)$ are real. In particular, $p_g(\alpha) > 0$. We define

$$q_m(\alpha) = -\frac{\overline{\eta_m}\sqrt{k^{\alpha+\mu}(0)}k^{\beta_m+\mu}(\zeta_{m+1})}{\prod_{k=1}^{m} b_{\mathbf{c}_k}(0)b(\zeta_{m+1})k^{\beta_m+\mu}(0)\sqrt{k^{\beta_m}_{\zeta_{m+1}}(\zeta_{m+1})}}$$

Let $n > m$. Note that $\mathfrak{z} f_n^\alpha$ has a simple poles at $\mathfrak{z}^{-1}(\infty)$. Therefore,

$$\mathfrak{z} f_n^\alpha - \frac{(\mathfrak{z} f_n^\alpha b)(0)k^{\alpha+\mu}}{k^{\alpha+\mu}(0)b} \in H^2(\alpha).$$

Using this and the fact that $(\mathfrak{z} f_n^\alpha)(\zeta_{m+1}) = 0$, we obtain

$$\begin{aligned}\langle \mathfrak{z} f_n^\alpha, f_m^\alpha\rangle = &-\frac{\eta_n\overline{\eta_m}}{\sqrt{k^{\beta_n}_{\zeta_{n+1}}(\zeta_{n+1})}\sqrt{k^{\beta_m}_{\zeta_{m+1}}(\zeta_{m+1})}}\\ &\times \frac{(\mathfrak{z} b)(0)\prod_{k=m+1}^{n} b_{\mathbf{c}_k}(0)k^{\beta_n}_{\zeta_{n+1}}(0)}{k^{\beta_m+\mu}(0)}\frac{k^{\beta_m+\mu}(\zeta_{m+1})}{b(\zeta_{m+1})} = p_n(\alpha)q_m(\alpha),\end{aligned}$$

by definition. Using in addition that $b(\bar{\zeta}) = \overline{b(\zeta)}$ we obtain in the same way as before that

$$\frac{\overline{k^{\beta_m+\mu}(\zeta_{m+1},0)}}{\overline{b(\zeta_{m+1})}} = \frac{\overline{\eta_m}^2 k^{\beta_m+\mu}(\zeta_{m+1},0)}{b(\zeta_{m+1})}.$$

Hence $q_m(\alpha) \in \mathbb{R}$. Finally, we consider the diagonal terms, i.e.,

$$\langle \mathfrak{z} f_n^\alpha, f_n^\alpha\rangle = \mathbf{c}_{n+1} - \frac{(\mathfrak{z} b)(0)k^{\beta_n}_{\zeta_{n+1}}(0)}{k^{\beta_n}_{\zeta_{n+1}}(\zeta_{n+1})k^{\beta_n+\mu}(0)}\frac{k^{\beta_n+\mu}(\zeta_{n+1})}{b(\zeta_{n+1})} = \mathbf{c}_{n+1} + p_n(\alpha)q_n(\alpha).$$

The structure of the resolvents given in Definition 2.4 follows by the conformal invariance of the Ahlfors function. More specifically, if $w = w_j = \frac{1}{\mathbf{c}_j - z}$, then $\Psi_j(w) := \Psi(z)$ is the Ahlfors function in the w-plane. The given ordering $\mathbf{C}$ generates the specific ordering

$$\mathbf{C}_j = \left\{\frac{1}{\mathbf{c}_{j+1}-\mathbf{c}_j}, \dots, \frac{1}{\mathbf{c}_g - \mathbf{c}_j}, 0, \frac{1}{\mathbf{c}_1 - \mathbf{c}_j}, \dots, \frac{1}{\mathbf{c}_{j-1}-\mathbf{c}_j}\right\}$$

and the multiplication by w is again a periodic GMP matrix (up to an appropriate shift). That is,

$$S^{-j}(\mathbf{c}_j - A(\alpha, \mathbf{C}))^{-1}S^j \in A(E_j, \mathbf{C}_j),$$

where $E_j = \{y = \frac{1}{c_j - x} \colon x \in E\}$. This shows that $A(\alpha) \in \mathrm{GMP}(\mathbf{C})$. Hence, $A(\alpha) \in A(E)$.

Now, we turn to the map $A(E, \mathbf{C}) \to \Gamma^*$. To $A \in A(E, \mathbf{C})$, we associate the resolvent functions $a_0^2 r_+$ and r_-^{-1}. Due to (3.4), Lemma 3.6 and Theorem 2.10, A is reflectionless on E. Hence, we can apply the construction of Section 1.2.4 to obtain $\alpha \in \Gamma^*$. Due to uniqueness of the associated character α, this map is one-to-one, up to the identification $(p_k, q_k) \mapsto (-p_k, -q_k)$. □

3.2.1 The magic formula and parametrization of $A(E, \mathbf{C})$

Let us turn to the proof of the magic formula and Theorem 2.12. The following lemma describes the coefficients of Δ^A in terms of A.

Lemma 3.10. *Let $A \in \mathrm{GMP}(\mathbf{C})$ be a periodic GMP matrix with generating coefficients $\vec{\mathbf{p}}$ and $\Delta^A(z)$ its discriminant, cf. (2.13). Then Δ^A is a rational function with simple poles at $\mathbf{c}_k$ and infinity, i.e.,*

$$\Delta^A(z) = d_0 + \nu_0 z + \sum_{k=1}^{g} \frac{\nu_k}{z - \mathbf{c}_k}.$$

Moreover, the coefficients are given by

$$\nu_0 = \frac{1}{p_g}, \qquad d_0 = -q_g - \nu_0 \sum_{j=1}^{g-1} p_j q_j, \qquad \nu_k = \Lambda_k(\vec{\mathbf{p}}) \qquad \text{for} \quad k = 1, \dots, g,$$

where Λ_k is defined in (2.14).

Proof. Considering the residues of Δ_A at $\mathbf{c}_k$ and infinity we obtain the coefficients ν_k, for $k = 0, \dots, g$. Thus it remains to show the formula for d_0. To this end, we write

$$\mathfrak{A}(z) = Az + B + \sum_{j=1}^{g} \frac{1}{\mathbf{c}_j - z} C_j,$$

where the constant term is given by

$$B = \begin{bmatrix} 0 & -p_g \\ \frac{1}{p_g} & -q_g \end{bmatrix} - \frac{1}{p_g} \sum_{j=1}^{g} \begin{bmatrix} 0 & p_{j-1}^2 \\ 0 & p_{j-1} q_{j-1} \end{bmatrix}$$

Since $\Delta^A = \operatorname{tr} \mathfrak{A}$, we also obtain the expression for d_0. □

Remark 3.11. Note that due to the definition of GMP matrices $\nu_k > 0$ for $k = 0, \dots, g$. Thus, Δ^A maps the upper half plane into itself.

Lemma 3.12. *Let E be a finite gap set and Δ_E the corresponding function from Lemma 2.2. Let $A \in \mathrm{GMP}(\mathbf{C})$ be periodic. Then the following are equivalent:*

(*i*) $A \in A(E, \mathbf{C})$,

(*ii*) $\Delta^A(z) = \Delta_E(z)$,

(*iii*) $\Delta^A(A) = \Delta_E(A)$.

Proof. Let $\Delta^A(z) = \Delta_E(z)$. By Theorem 2.10, we obtain that

$$\sigma(A) = \left\{ z \in \mathbb{C} \colon \Delta^A(z) \in [-2, 2] \right\} = \{ z \in \mathbb{C} \colon \Delta_E(z) \in [-2, 2] \} = E.$$

Thus, $A \in A(E, \mathbf{C})$. On the other hand, if $A \in A(E, \mathbf{C})$, then $\{ z \in \mathbb{C} \colon \Delta^A(z) \in [-2, 2] \} = E$. By the previous remark and the uniqueness of Δ_E, we obtain $\Delta^A(z) = \Delta_E(z)$. Hence, (*i*) $\iff$ (*ii*). (*ii*) $\implies$ (*iii*) is clear. On the last non-zero diagonal of $\Delta^A(A)$, i.e., $\Delta^A(A)_{j,g+1+j}$ for $j = 0, \dots, g$, only one of the summands is non-vanishing. With the notation from the previous lemma, (*iii*) yields

$$(\nu_k - \lambda_k)(\mathbf{c}_k - A)^{-1}_{k-1,g+k} = 0, \qquad (\lambda_0 - \nu_0) A_{g,2g+1} = 0.$$

Hence, $\Delta^A(z) = \Delta_E(z)$. □

Proof of Theorem 2.6. Let $A \in A(E, \mathbf{C})$. Due to Proposition 2.5, there exists $\alpha \in \Gamma$ such that $A = A(\alpha)$. Hence, (2.5) is the magic formula in terms of functional models.

Let $A \in \mathrm{GMP}(\mathbf{C})$ satisfy $\Delta_E(A) = S^{g+1} + S^{-(g+1)}$. Naĭman's lemma (cf. [56, Lemma 8.2.4]) implies that A is periodic. Since by definition $A \in A(\sigma(A), \mathbf{C})$, we obtain that

$$\Delta^A(A) = S^{g+1} + S^{-(g+1)} = \Delta_E(A).$$

By Lemma 3.12 $A \in A(E, \mathbf{C})$. □

Proof of Theorem 2.12. Let $A \in A(E, \mathbf{C})$. Due to Lemma 3.12, $\Delta^A(z) = \Delta_E(z)$ and by Lemma 3.10, the coefficients of A satisfy (2.15). On the other hand, if the coefficients satisfy (2.15), then $\Delta^A(z) = \Delta_E(z)$ and hence $A \in A(E, \mathbf{C})$. □

The Jacobi flow, given in Definition 2.7, allows to iteratively read out the coefficients of $J(\alpha)$. There is an obvious map $\mathcal{F}\colon A(E) \to J(E)$ defined by

$$\mathcal{F}A(\alpha) = J(\alpha).$$

The question is, how to find the coefficients of $J(\alpha)$ in terms of coefficients of $A(\alpha)$?

Proposition 3.13. *To $\alpha \in \Gamma^*$, we associate a* GMP *matrix $A(\alpha)$ with coefficients $(\vec{p}(\alpha), \vec{q}(\alpha))$ and a Jacobi matrix $J(\alpha)$ with coefficients $(a_j(\alpha), b_j(\alpha))$. Then we have*

$$a_0(\alpha) = \|\vec{p}(\alpha)\| \qquad \text{and} \qquad b_{-1}(\alpha) = p_g(\alpha) q_g(\alpha).$$

Proof. Let $P_+(\alpha)$ be the orthogonal projection onto $H^2(\alpha)$. Since $f^\alpha_{-1} = e^\alpha_{-1}$, we see that

$$a_0(\alpha) = \|P_+(\alpha) \mathfrak{z} e^\alpha_{-1}\| = \|P_+(\alpha) \mathfrak{z} f^\alpha_{-1}\| = \|p(\alpha)\|$$

and

$$b_{-1}(\alpha) = \langle e^\alpha_{-1}, \mathfrak{z} e^\alpha_{-1}\rangle = \langle f^\alpha_{-1}, \mathfrak{z} f^\alpha_{-1}\rangle = p_g(\alpha) q_g(\alpha). \qquad \square$$

Chapter 4

Asymptotics of L^∞ extremal polynomials

4.1 Asymptotics of the Chebyshev polynomials of a circular arc

We present the proofs to the results of Section 2.2.1.

We choose the standard normalization for the complex Green's function $B_{\Omega_\alpha}(\infty,\zeta_0)>0$, if $\zeta_0\neq\infty$, $(\zeta B_{\Omega_\alpha}(\zeta,\infty))(\infty)>0$ Moreover, we assume that the extremal polynomial meets the normalization $B_{\Omega_\alpha}(\zeta_0,\infty)^n P_{n,\zeta_0}(\zeta_0)>0$. Let $\mathcal{P}_n$ be the set of all polynomials of degree at most n and

$$\mathcal{P}_n(E_\alpha)=\{P\in\mathcal{P}_n:\|P\|_{E_\alpha}\leq 1\}. \tag{4.1}$$

Since $E_\alpha\subset\mathbb{T}$, the map

$$P_n(\zeta)\mapsto P_n^*(\zeta):=\zeta^n\overline{P_n\left(\zeta^*\right)},\quad \zeta^*=1/\overline{\zeta},$$

is an involution on $\mathcal{P}_n(E_\alpha)$. This shows that for $L_n(\infty):=1/\|T_n\|_{E_\alpha}$ we have

$$L_n(\infty)=L_n(0) \tag{4.2}$$

and there exists $\phi\in\mathbb{R}$ such that $P_{n,\infty}=e^{i\phi}P_{n,0}^*$. We will give a solution of (4.2) by reducing it to a problem which was already considered by Yuditskii [70]. Let $E_0=\mathbb{R}\setminus(-1,1)$ and $\Omega_0=\mathbb{C}\setminus E_0$. The map

$$\zeta(z)=\frac{z-z_0}{z-\overline{z}_0},\qquad z_0=i\tan(\alpha/2),$$

maps Ω_0 conformally onto Ω_α. By $z(\zeta)$ we denote its inverse map. Henceforth, if we use z and ζ simultaneously we have in mind $z(\zeta)$ and $\zeta(z)$, respectively. Defining $z_\infty=z(\infty)$ it is obvious that $z_\infty=\overline{z}_0$. To the polynomial $E_n(z)=(z-z_\infty)^n$ we associate the weighted norm

$$\|Q_n\|_{\Pi(E_n)}:=\sup_{x\in E_0}\left|\frac{Q_n(x)}{E_n(x)}\right|\quad\text{for } Q_n\in\mathcal{P}_n.$$

Lemma 4.1. *Let $Q_{n,z_0} \in \mathcal{P}_n$ be the solution of the extremal problem*

$$|Q_{n,z_0}(z_0)| = \sup\{|Q_n(z_0)| : \ Q_n \in \mathcal{P}_n, \|Q_n\|_{\Pi(E_n)} \le 1\}.$$

Then

$$P_{n,0}(\zeta) = e^{i\phi} \frac{Q_{n,z_0}(z)}{E_n(z)}, \tag{4.3}$$

where ϕ is a proper phase.

Proof. We have

$$C \frac{z - z_l}{z - z_\infty} = \zeta(z) - \zeta(z_l), \quad C = \frac{z_0 - z_\infty}{z_l - z_\infty}.$$

Hence, the map

$$P_n(\zeta) \mapsto Q_n(z) := E_n(z) P_n(\zeta(z)),$$

maps $\mathcal{P}_n$ bijectively onto itself. Moreover,

$$\|P_n\|_{E_\alpha} = \sup_{x \in E_0} \left| \frac{Q_n(x)}{E_n(x)} \right| = \|Q_n\|_{\Pi(E_n)}.$$

Therefore,

$$\begin{aligned} |P_{n,0}(0)| &= \sup\{|P_n(0)| : \ P_n \in \mathcal{P}_n(E_\alpha)\} \\ &= \sup \left\{ \left| \frac{Q_n(z_0)}{E_n(z_0)} \right| : \ Q_n \in \mathcal{P}_n, \|Q_n\|_{\Pi(E_n)} \le 1 \right\} = \left| \frac{Q_{n,z_0}(z_0)}{E_n(z_0)} \right| \end{aligned}$$

and (4.3) holds. □

In [70] an explicit solution for this kind of problem is given. First, let us mention that in [62] it is shown that for fixed α there may be $N \in \mathbb{N}$ such that for $n < N$ the extremal polynomial is just z^n. This corresponds to a special case in [70]. Since we are only interested in asymptotics we assume that $n > N$. We recall the theorem in a way that is convenient for our purpose. Let $\omega(z, I; \Omega)$ denote the harmonic measure of the domain Ω.

Theorem 4.2 ([70])**.** *Let $E_n(z)$ be a polynomial with zeros $Z = \{\overline{z}_1, \dots, \overline{z}_n\} \subset \mathbb{C}_-$ and $z_0 \in i\mathbb{R}_{>0}$. Then there exists a unique $0 < x_n < 1$ such that $I_n = [-x_n, x_n]$ satisfies*

$$\sum_{\overline{z}_l \in Z \cup \{\overline{z}_0\}} \omega(\overline{z}_l, I_n; \Omega_0 \setminus I_n) = 1. \tag{4.4}$$

Let $\Omega_n = \Omega_0 \setminus I_n$ and set

$$\mathfrak{B}(z) = \prod_{\overline{z}_l \in Z \cup \{\overline{z}_0\}} B_{\Omega_n}(z, \overline{z}_l), \ \ s_n(z) = \sqrt{\frac{z_0^2 - x_n^2}{z_0^2 - 1} \frac{z^2 - 1}{z^2 - x_n^2}}, \quad s_n(z_0) = 1.$$

The extremal polynomial Q_{n,z_0} is up to the unimodular factor $e^{i\phi}$ uniquely given by

$$Q_{n,z_0}(z) = e^{i\phi} E_n(z) \left(\frac{1 + s_n(z)}{2} \frac{1}{\mathfrak{B}(z)} + \frac{1 - s_n(z)}{2} \frac{z - z_0}{z - \overline{z}_0} \frac{\overline{E_n(\overline{z})}}{E_n(z)} \mathfrak{B}(z) \right).$$

Hence, by Theorem 4.2 the solution of the extremal problem is given by

$$Q_{n,z_0}(z) = E_n(z)\left(\frac{1+s_n(z)}{2}\frac{1}{B_n(z)^{n+1}} + \frac{1-s_n(z)}{2}\frac{(z-z_0)^{n+1}}{(z-\overline{z}_0)^{n+1}}B_n(z)^{n+1}\right),$$

where $B_n(z) = B_{\Omega_n}(z, z_\infty)$. We also abbreviate $G_n(z) = G_{\Omega_n}(z, z_\infty)$, $G(z) = G_{\Omega_0}(z, z_\infty)$, $B(z) = B_{\Omega_0}(z, z_\infty)$ and $\omega_n(I) = \omega(z_\infty, I; \Omega_n)$. Note that (4.4) reads

$$\omega_n(I_n) = \frac{1}{n+1} \tag{4.5}$$

in this case. Our goal is to find the limit of $B_{\Omega_\alpha}(\zeta, \infty)^n P_{n,0}(\zeta)$. Due to the conformal invariance of the Green's function, this is equivalent to finding the asymptotics of

$$f_n(z) = \frac{B(z)^n Q_{n,z_0}(z)}{E_n(z)}.$$

By the maximum principle and Montel's theorem, there exist subsequences n_j such that f_{n_j} converges to an analytic function f uniformly on compact subsets of Ω_0. We will show that all subsequences have the same limit.

Lemma 4.3. *Let $I_n = [-x_n, x_n]$. Then $x_n \to 0$ as $n \to \infty$ and*

$$\lim_n s_n(z)^2 = \frac{z_0^2}{z_0^2-1}\frac{z^2-1}{z^2},$$
$$\lim_n \frac{1-s_n(z)}{2}\frac{(z-z_0)^{n+1}}{(z-\overline{z}_0)^{n+1}}B(z)^n B_n(z)^{n+1} = 0,$$

uniformly on compact subsets of $\Omega_0 \setminus \{0\}$.

Proof. By the maximum principle (see [47, Ch. IV, Sec 2]) $\omega_n(I_n)$ is an increasing function of x_n. Since $\omega_n(I_n) \to 0$ as $x_n \to 0$, we obtain the first statement and the second statement is clear. Since

$$\left|\frac{(z-z_0)}{(z-\overline{z}_0)}B_n(z)B(z)\right| < 1 \quad \text{on } \Omega_n$$

we obtain the last statement. □

Lemma 4.4. *Uniformly on compact subsets of Ω_α*

$$\lim_{n\to\infty} B_{\Omega_\alpha}(\zeta, \infty)^n P_{n,\infty}(\zeta) = \frac{1-s(z)}{2}\frac{B_{\Omega_0}(z,0)}{B_{\Omega_0}(z,z_0)},$$

where

$$s(z) = \frac{z_0}{\sqrt{z_0^2-1}}\frac{\sqrt{z^2-1}}{z}.$$

Proof. Solving the Dirichlet problem for the harmonic function

$$h(z_1) = G_{\Omega_0}(z_1, z) - G_{\Omega_n}(z_1, z)$$

in Ω_n shows

$$h(z_1) = \int_{I_n} G_{\Omega_0}(z, x)\omega(z_1, \mathrm{d}x; \Omega_n).$$

The symmetry of the Green's function with respect to the variables z and z_1 leads to

$$G_{\Omega_0}(z, z_1) - G_{\Omega_n}(z, z_1) = \int_{I_n} G_{\Omega_0}(z, x)\omega(z_1, \mathrm{d}x; \Omega_n).$$

Therefore,

$$\log\left|\frac{B(z)^n}{B_n(z)^n}\right| = n(G_n(z) - G(z)) = -\int_{I_n} G_{\Omega_0}(z, x) n\omega_n(\mathrm{d}x).$$

By (4.5), $\chi_{I_n} n\omega_n(\mathrm{d}x)$ converges to the delta distribution and therefore

$$\lim_{n\to\infty} \log\left|\frac{B(z)^n}{B_n(z)^n}\right| = -G_{\Omega_0}(z, 0).$$

In the same way we see that $\lim_{n\to\infty} \log|B_{\Omega_n}(z, \overline{z}_0)| = -G_{\Omega_0}(z, \overline{z}_0)$. Therefore, in combination with Lemma 4.3 we obtain for the limit function f that

$$|f(z)| = \left|\frac{1 + s(z)}{2}\frac{B_{\Omega_0}(z, 0)}{B_{\Omega_0}(z, \overline{z}_0)}\right|,$$

and hence

$$f(z) = \frac{1 + s(z)}{2}\frac{B_{\Omega_0}(z, 0)}{B_{\Omega_0}(z, \overline{z}_0)}.$$

This shows

$$\lim_{n\to\infty} \frac{B(z)^n Q_{n,z_0}(z)}{E_n(z)} = e^{i\phi}\frac{1 + s(z)}{2}\frac{B_{\Omega_0}(z, 0)}{B_{\Omega_0}(z, \overline{z}_0)}.$$

Due to the symmetry of the domain with respect ot the real line we have $\overline{B_{\Omega_\alpha}(\overline{\zeta}, \infty)} = B_{\Omega_\alpha}(\zeta, \infty)$. This and the conformal invariance of the Green's function lead to

$$\begin{aligned} B_{\Omega_\alpha}(\zeta, \infty)^n P_{n,\infty}(\zeta) &= B_{\Omega_\alpha}(\zeta, \infty)^n e^{i\phi}\zeta^n \overline{P_{n,0}(1/\overline{\zeta})} \\ &= e^{i\phi} B_{\Omega_\alpha}(\zeta, 0)^n \overline{P_{n,0}(\zeta^*)} \\ &= e^{i\phi} \overline{B_{\Omega_\alpha}(\zeta^*, \infty)^n P_{n,0}(\zeta^*)}. \end{aligned}$$

Since $\zeta^*(z) = \zeta(\overline{z})$, the relation $s(z) = -\overline{s(\overline{z})}$ almost concludes the proof. Evaluating the identity at infinity shows that $\phi = 0$. □

The next natural question is to solve this problem not only for $\zeta_0 = 0$, but for an arbitrary point $\zeta_0 \in \Omega_\alpha$. As before, due to the symmetry of the domain, we can reduce it to $\zeta_0 \in \mathbb{D}$. Namely, if $|\zeta_0| > 1$, we have

$$P_{n,\zeta_0} = e^{i\phi} P^*_{n,\zeta_0^*}, \quad B_{\Omega_\alpha}(\zeta, \infty)^n P_{n,\zeta_0}(\zeta) = \overline{B_{\Omega_\alpha}(\zeta^*, \infty)^n P_{n,\zeta_0^*}(\zeta^*)}. \tag{4.6}$$

Lemma 4.5. *Let $|\zeta_0| < 1$ and $z_{\zeta_0} = z(\zeta_0)$. Let K_0 be the unique circle that passes through z_{ζ_0} and $\overline{z}_{\zeta_0}$ such that Ω_0 is symmetric with respect to reflection by K_0. Moreover, let $\{x_0\} = K_0 \cap (-1,1)$ and*

$$s(z, z_{\zeta_0}) = \frac{z_{\zeta_0} - x_0}{\sqrt{z_{\zeta_0}^2 - 1}} \frac{\sqrt{z^2-1}}{z - x_0}.$$

Then uniformly on compact subsets of Ω_α

$$\lim_{n\to\infty} B_{\Omega_\alpha}(\zeta,\infty)^n P_{n,\zeta_0}(\zeta) = e^{i\phi} \frac{1 + s(z,z_{\zeta_0})}{2} \frac{B_{\Omega_0}(z,x_0)}{B_{\Omega_0}(z,\overline{z}_{\zeta_0})},$$

where ϕ is a proper phase.

Proof. Let $\zeta_0 \in \mathbb{D}$. By a Möbius transformation ψ, (aka Blaschke factor of the disc) we map $\zeta_0 \mapsto 0$ such that E_α is mapped onto $E_{\alpha'}$ for some α', i.e., $E_{\alpha'}$ is still symmetric with respect to the real axis and $1 \in E_{\alpha'}$. Then we compose this map with z (related to α') of the previous section in order to obtain a conformal map $\tilde{z} : \Omega_\alpha \to \Omega_0$ such that

$$\tilde{z}(e^{i\alpha}) = -1, \quad \tilde{z}(e^{-i\alpha}) = 1, \quad \tilde{z}(\zeta_0) = i\tan(\alpha'/2).$$

Hence, we can apply exactly the same procedure in proving the asymptotics

$$\lim_j \frac{B_{\Omega_0}(\tilde{z},\tilde{z}_\infty)^{n_j} \tilde{Q}_{n_j,\tilde{z}_0}(\tilde{z})}{\tilde{E}_{n_j}(\tilde{z})} = \frac{1+\tilde{s}(\tilde{z})}{2} \frac{B_{\Omega_0}(\tilde{z},0)}{B_{\Omega_0}(\tilde{z},\overline{\tilde{z}(\zeta_0)})},$$

where

$$\tilde{s}(\tilde{z})^2 = \frac{\tilde{z}(\zeta_0)^2}{\tilde{z}(\zeta_0)^2 - 1} \frac{\tilde{z}^2 - 1}{\tilde{z}^2}.$$

The map $\phi : \Omega_0 \to \Omega_0$ with $\phi(\tilde{z}) = z$ is a fractional linear transformation (FLT) with $\phi(\mathbb{R}) = \mathbb{R}$.

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus E_\alpha & \xrightarrow{\ z\ } & \Omega_0 \\ \downarrow \psi & & \uparrow \phi \\ \overline{\mathbb{C}} \setminus E_{\alpha'} & \xrightarrow{\ \tilde{z}\ } & \Omega_0 \end{array}$$

Due to properties of conformal maps in particular of FLTs we obtain $\phi(i\mathbb{R}) = K_0$, $\phi(0) = x_0$, $\phi(\tilde{z}(\zeta_0)) = z_{\zeta_0}$ and $\phi\left(\overline{\tilde{z}(\zeta_0)}\right) = \overline{z}_{\zeta_0}$, which concludes the proof. □

Proof of Theorem 2.13 and Theorem 2.14. This follows directly by the two preceding lemmata and the conformal invariance of the complex Green's function. □

We define

$$L(\zeta) := \lim_n e^{-nG_{\Omega_\alpha}(\zeta,\infty)} L_n(\zeta).$$

Note that (4.6) in particular implies that $L(\zeta_0^*) = L(\zeta_0)$.

Let us point out that the fact that we don't give a formula for P_{n,ζ_0} for $|\zeta_0| = 1$ is just a consequence of our technique. Indeed, a solution for $|\zeta_0| = 1$ could be obtained in the same way as it was demonstrated in Section 4.2.4. We only use that for each n there exists a maximizer P_{n,ζ_0} (either by referring to the real Chebyshev problem or by compactness of $\mathcal{P}_n(E_\alpha)$). Due to Montel's theorem, $B_{\Omega_\alpha}(\zeta,\infty)^n P_{n,\zeta_0}(\zeta)$ has a convergent subsequence, i.e., there exists $f(\zeta)$ such that $f(\zeta) = \lim_j B_{\Omega_\alpha}(\zeta,\infty)^{n_j} P_{n_j,\zeta_0}(\zeta)$. Set $L(\zeta_0) = f(\zeta_0)$. We will see that with this definition $L(\zeta)$ is continuous and therefore this value is independent of the particular choice of the subsequence.

Lemma 4.6. *$L_n(\zeta)$ and $L(\zeta)$ are continuous on $\mathbb{C}\setminus E_\alpha$ and Ω_α, respectively.*

Proof. Let $P \in \mathcal{P}_n(E_\alpha)$. Since $|B_{\Omega_\alpha}(\zeta,\infty)^n P(\zeta)| \leq 1$, P is locally bounded and therefore $\mathcal{P}_n(E_\alpha)$ is equicontinuous. Hence, for every $\zeta_0 \in \mathbb{C}\setminus E_\alpha$ there exists $\delta > 0$ such that $|\zeta - \zeta_0| < \delta$ implies

$$L_n(\zeta_0) \geq |P_{n,\zeta}(\zeta_0)| > |P_{n,\zeta}(\zeta)| - \epsilon = L_n(\zeta) - \epsilon.$$

In the same way we see that $L_n(\zeta) > L_n(\zeta_0) - \epsilon$ and therefore $|L_n(\zeta) - L_n(\zeta_0)| < \epsilon$. The same proof applies for $L(\zeta)$. □

4.1.1 Log-Subharmonicity and Reproducing Kernels

In this chapter we will prove some properties of the extremal values $L_n(\zeta)$ and $L(\zeta)$ as functions on Ω_α. We recall the definition of log-subharmonicity.

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{R}$ be an upper semicontinuous function. It is called subharmonic if for every $z_0 \in \Omega$ there exists R such that $\{z : \ |z - z_0| \leq R\} \subset \Omega$ and for all $0 < r \leq R$ we have

$$f(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\mathrm{d}t.$$

A function is called log-subharmonic if $\log f$ is subharmonic.

Remark 4.7. If f is twice continuously differentiable, then f is subharmonic if and only if $\Delta f \geq 0$ in Ω.

Proposition 4.8. *$L_n(\zeta)$ and $L(\zeta)$ are* log *subharmonic on $\mathbb{C}\setminus E_\alpha$ and Ω_α, respectively.*

Proof. The modulus of an analytic function is log subharmonic. Since L_n is continuous it can be easily seen that it is log subharmonic as the upper envelope of polynomials; cf. [41, Lecture 7] or [53, Theorem 3.4.2]. Clearly, this also holds for $|B_{\Omega_\alpha}(\zeta,\infty)|^n L_n(\zeta)$. Note that

$$\log|B_{\Omega_\alpha}(\zeta,\infty)^n L_n(\zeta)| = n(G_n(z) - G(z)) + G_{\Omega_n}(z, \overline{z}_{\zeta_0})$$

By the maximum principle $G_{\Omega_n}(z, z_1)$ is increasing in n and therefore this holds for $n(G_n(z) - G(z)) + G_{\Omega_n}(z, \overline{z}_{\zeta_0})$. Thus, we can interchange limit and integration and obtain that $L(\zeta)$ is log subharmonic. □

Proof of Theorem 2.15. The function λ given by (2.21) is a composition of the maps $z : \Omega_\alpha \to \Omega_0$, $w : \Omega_0 \to \mathbb{C}_+$, defined by $w(z) = \sqrt{\frac{z-1}{z+1}}$ and $\tilde{\lambda}(w) : \mathbb{C}_+ \to \mathbb{H}_+$ defined by $\lambda(w) = -iw$. Let $w_0 = w(z_{\zeta_0})$. Using the reflection principle and that FLTs map circles onto circles, we obtain that $w(x_0) = i|w_0|$ and $w(\overline{z}_{\zeta_0}) = -\overline{w}_0$. Evaluating (2.20) at ζ_0, we obtain

$$\begin{aligned} L(\zeta_0) &= \left|\frac{\lambda_0 - |\lambda_0|}{\lambda_0 + |\lambda_0|}\right| \frac{2|\lambda_0|}{|\lambda_0 - \overline{\lambda_0}|} \\ &= \frac{|(\lambda_0 - |\lambda_0|)(\overline{\lambda_0} + |\lambda_0|)|}{|\lambda_0 + |\lambda_0||^2} \frac{2|\lambda_0|}{|\lambda_0 - \overline{\lambda_0}|} \\ &= \frac{|\lambda_0||\lambda_0 - \overline{\lambda_0}|}{|\lambda_0||\sqrt{\lambda_0} + \overline{\sqrt{\lambda_0}}|^2} \frac{2|\lambda_0|}{|\lambda_0 - \overline{\lambda_0}|} \\ &= \frac{2|\lambda_0|}{|\sqrt{\lambda_0} + \overline{\sqrt{\lambda_0}}|^2} = k_{Ch}(\lambda_0, \lambda_0). \end{aligned}$$

□

Remark 4.9. (i) Let $\partial, \overline{\partial}$ denote the Wirtinger derivatives. Since $4\partial\overline{\partial} = \Delta$, we have for twice continuously differentiable functions that

$$f \text{ is } \log\text{subharmonic} \iff \begin{bmatrix} f(z) & \overline{\partial} f(z) \\ \partial f(z) & \partial\overline{\partial} f(z) \end{bmatrix} \geq 0.$$

Note that this matrix-inequality, which appears naturally for L as limit of an upper envelope of polynomials is just a small part of an matrix-inequality, which holds for reproducing kernels of analytic functions. Namely, writing

$$k(z, z_0) = \sum_k \phi_k(z)\overline{\phi_k(z_0)},$$

for an orthonormal basis $\{\phi_k\}$, we see that the matrix $\{\partial^i\overline{\partial}^j k(z_0, z_0)\}_{i,j=1}^n$ is the Gram matrix of the vectors $\{\phi_k(z_0)\}, \{\partial\phi_k(z_0)\}, \ldots, \{\partial^n\phi_k(z_0)\}$ with respect to the standard ℓ^2 scalar product and therefore

$$\{\partial^i\overline{\partial}^j k(z_0, z_0)\}_{i,j=1}^n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

As a corollary of Theorem 2.13, we obtain (2.18). Recall that

$$\mathrm{Cap}(E_\alpha) := |\zeta B_{\Omega_\alpha}(\zeta, \infty)|(\infty).$$

Corollary 4.10. *Let T_n denote the Chebyshev polynomials of E_α. Then*

$$\lim_{n\to\infty} \frac{\|T_n\|_{E_\alpha}}{\mathrm{Cap}(E_\alpha)^n} = 2\cos(\alpha/4)^2.$$

Proof. By definition we have

$$|B_{\Omega_\alpha}(\zeta, \infty)^n L_n(\zeta)|(\infty) = \frac{\mathrm{Cap}(E_\alpha)^n}{\|T_n\|_{E_\alpha}}.$$

Since $\lambda(\infty) = e^{i\frac{\alpha}{2}}$, we see that

$$k_{\Omega_\alpha}(\infty, \infty) = \frac{1}{2\cos(\alpha/4)^2}.$$

□

4.2 Polynomial Ahlfors Problem

This section is related to the results presented in Section 2.2.2.

4.2.1 Comb-domains and potential theory

We start by extending the relation of complex Green's function and conformal mappings onto corresponding comb-domains, which was discussed for the set E_J already in the preliminary chapter, also for the sets E_T and E_S. These domains are standard objects in the spectral theory of reflectionless operators [43], see also [19, 25].

$$\begin{aligned}
\Pi_J =&\{\vartheta = \xi + i\eta : 0 < \xi < \pi,\ \eta > 0\} \setminus \cup_{j=1}^{g}\{\vartheta = \omega_j + i\eta, \eta \in (0, h_j]\},\\
\Pi_T =&\mathbb{C}_+ \setminus \cup_{j=0}^{g} \cup_{m\in\mathbb{Z}} \{\vartheta = \omega_j + 2\pi m + i\eta, \eta \in (0, h_j]\},\\
\Pi_S =&\{\vartheta = \xi + i\eta : \xi > 0,\ \eta > 0\} \setminus \cup_{j=1}^{g}\{\vartheta = \omega_j + i\eta, \eta \in (0, h_j]\}.
\end{aligned}$$

In the first case $\omega_j \in (0, \pi)$ in the second one $\omega_0 = 0$ and $\omega_j \in (0, 2\pi)$ for $j = 1, ..., g$. In the setting of reflectionless differential operators the values ω_j are often referred to as frequencies. We map conformally $\mathbb{C}_+$ onto one of the corresponding combs making normalizations

$$\theta_J(\mathbf{b}_0) = 0, \quad \theta_J(\mathbf{a}_0) = \pi, \quad \theta_J(\infty) = \infty, \tag{4.7}$$

$$\theta_T(iy) \sim iy,\ y \to \infty, \quad \theta_T(0) = 0, \tag{4.8}$$

$$\theta_S(-x) \sim i\sqrt{x},\ x \to \infty, \quad \theta_S(0) = 0. \tag{4.9}$$

Note that in the second case we automatically get that there exists a certain b such that $\theta_T(z + b) = \theta_T(z) + 2\pi$. In [19, Lemma A.2] it is shown that $b = 2\pi$, that is, $e^{i\theta_T}$ is well defined as a function of $\zeta = e^{iz} \in \mathbb{D}$. In the first and third case we get the gaps $(\mathbf{a}_j, \mathbf{b}_j)$ as preimages of the corresponding vertical slits, $j = 1, ..., g$. In the same way, in the second case we get a system of arcs $\{\zeta = e^{iz} : z \in (\mathbf{a}_j, \mathbf{b}_j)\}_{j=0}^{g}$, which form the complement of E_T.

If θ denotes one of the maps (4.7)–(4.9), then $\operatorname{Im}\theta(z)$ can be extended through the gaps to a single-valued harmonic function in Ω. Moreover, by expecting the behavior at the singularities we obtain as in Lemma 1.8

$$\operatorname{Im}\theta_J(z) = G_\Omega(z, \infty), \quad \operatorname{Im}\theta_T(z) = G_\Omega(e^{iz}, 0) + G_\Omega(e^{iz}, \infty), \quad \operatorname{Im}\theta_S(z) = M_\Omega(z), \tag{4.10}$$

where $G_\Omega(z, z_0)$ denotes the Green's function with respect to z_0 in the corresponding domain and $M_\Omega(z)$ stands for the Martin's function with respect to infinity. The function $e^{i\theta(z)}$ can be extended to Ω by the symmetry principle as a multi-valued function.

Let us fix Ω in the form $\mathbb{C} \setminus E_S$. Recall that for an arbitrary $\overline{\mathbb{C}} \setminus E_T$ or $\overline{\mathbb{C}} \setminus E_J$ we can always find a suitable conformally equivalent domain Ω of the above form. As already mention in Section 2.2.2 we will use $\mathbb{C}_+$ as covering domain at this place. Moreover, in contrary to the previous sections, in the notation we will not distinguish between the map and the variable. In the following we will show that the covering map $z = z(\lambda)$ also corresponds to a conformal mapping. For a given E_S there exists a system of half discs $\mathbb{D}_j^+$ such that the conformal mapping

$$\mathbb{C}_+ \to \mathcal{F}_+ = \{\lambda = \xi + i\eta : \xi > 0, \eta > 0\} \setminus \cup_{j=1}^{g}\mathbb{D}_j^+, \quad \lambda(-x) \simeq i\sqrt{x},\ x \to \infty,$$

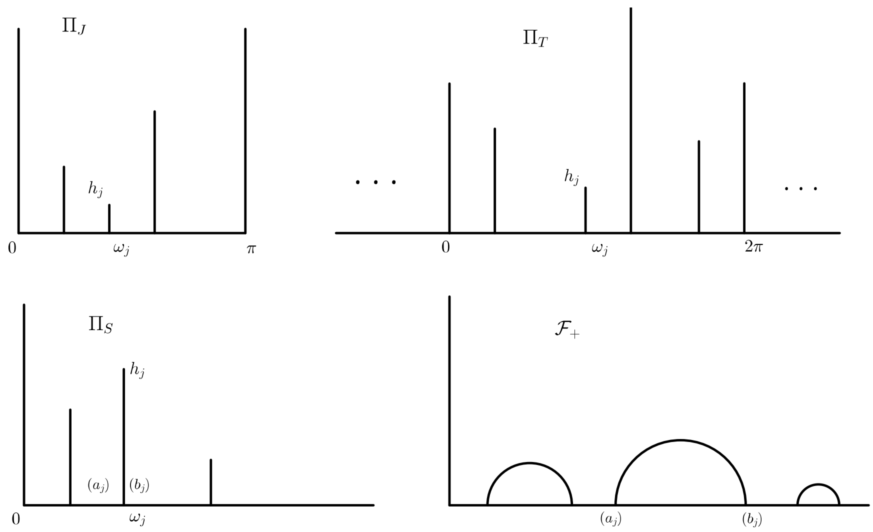


Figure 4.1: Comb-domains and the uniformization plane

transforms the negative half axis into the imaginary half axis in the λ-plane and the gaps $(\mathbf{a}_j, \mathbf{b}_j)$ into the boundary of the half discs $\partial\mathbb{D}_j^+$, see Fig. 4.1. The inverse map $z(\lambda)$ can be extended by a system of reflections to the whole upper half-plane. Indeed, let γ_j, acting in the λ-plane corresponds to the double reflection with respect to the negative half axis and the gap $(\mathbf{a}_j, \mathbf{b}_j)$ in the z-plane. This is a fractional linear transformation, which maps $\partial\mathbb{D}_j^+$ on $-\overline{\partial\mathbb{D}_j^+}$ (where the bar means complex conjugation) and we have $z(\gamma_j(\lambda)) = z(\lambda)$. In this case, the system $\{\gamma_j\}_{j=1}^g$ represents a generator of the Fuchsian group Γ and $\mathcal{F} = \mathcal{F}_+ \cup \overline{-\mathcal{F}_+} \cup i\mathbb{R}_+$ is a fundamental domain for the action of Γ on $\mathbb{C}_+$ such that $\mathbb{C}_+/\Gamma \simeq \mathbb{C} \setminus E_S$ (respectively to $\overline{\mathbb{C}} \setminus E_J$ or $\overline{\mathbb{C}} \setminus E_T$).

We have already seen that conformal mappings on comb-domains are special cases of Schwarz-Christoffel transformations. Recall that in Lemma 1.8 we proved that

$$\theta_J(z) = i\int_{\mathbf{b}_0}^{z} \prod_{j=1}^{g} \frac{z - \mathbf{d}_j}{\sqrt{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}} \frac{\mathrm{d}z}{\sqrt{(z-\mathbf{a}_0)(z-\mathbf{b}_0)}},$$

where $\mathbf{d}_j \in (\mathbf{a}_j, \mathbf{b}_j)$ corresponds to the top $\omega_j + ih_j$ of the slit.

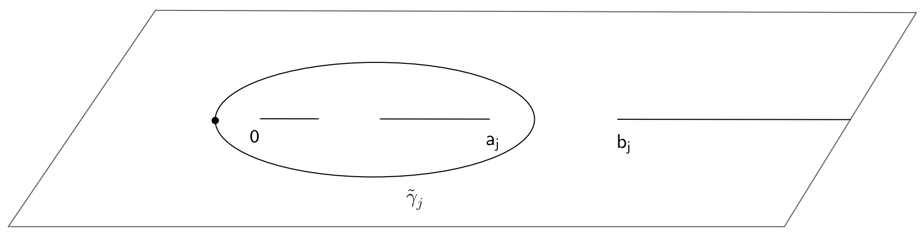


Figure 4.2: A generator of the fundamental group

Now we extend $e^{i\theta_S(z)}$ along the generator $\tilde{\gamma}_j$ of the fundamental group in Ω, see Fig. 4.2, which corresponds to the action of γ_j on the universal covering. As result we obtain

$$e^{i\theta_S(z(\gamma_j(\lambda))} = \alpha_S(\gamma_j)e^{i\theta_S(z(\lambda))}, \quad \alpha_S(\gamma_j) = e^{2\omega_j i}.$$

This system of multipliers given on the generators forms an element $\alpha_S(\gamma)$, $\gamma \in \Gamma$, of the group of *characters* Γ^*. This observation is in fact true for all three cases. In general let α_J, α_T and α_S denote the characters of the multi-valued functions $e^{i\theta_J}, e^{i\theta_T}$ and $e^{i\theta_S}$, respectively. As before, for $\lambda_0 \in \mathbb{C}_+$ and $z_0 = z(\lambda_0)$ let

$$b_{z_0}(\lambda) = \prod_{\gamma\in\Gamma} \frac{\lambda - \gamma(\lambda_0)}{\lambda - \overline{\gamma(\lambda_0)}} e^{i\psi_\gamma}, \quad e^{i\psi_\gamma} := \frac{i - \overline{\gamma(\lambda_0)}}{i - \gamma(\lambda_0)} \left| \frac{i - \gamma(\lambda_0)}{i - \overline{\gamma(\lambda_0)}} \right|$$

denote the Blaschke product with zeros at the full preimage $\theta^{-1}(\lambda_0)$. Recall the relation

$$-\log |b_{z_0}(\lambda)| = G_\Omega(z(\lambda), z_0).$$

The character generated by b_{z_0} is denoted by μ_{z_0}. In particular if $\Omega = \Omega_J$, we have that $b_\infty(\lambda) = e^{i\theta_J(z(\lambda))}$ and $\mu_\infty = \alpha_J$.

In the domain $\Omega_T = \overline{\mathbb{C}} \setminus E_T$ we have, see (4.10),

$$G_{\Omega_T}(\zeta(\lambda), 0) = \frac{-\log|\zeta(\lambda)| + \operatorname{Im} \theta_T(\zeta(\lambda))}{2}, \quad \zeta(\lambda) = \frac{b_0(\lambda)}{b_\infty(\lambda)}. \tag{4.11}$$

That is,

$$b_0(\lambda) = \sqrt{\zeta(\lambda)e^{i\theta_T(\zeta(\lambda))}} \quad \text{and} \quad \mu_0^2 = \alpha_T, \ \mu_0(\gamma_j) = \mu_\infty(\gamma_j) = e^{i\omega_j}.$$

Finally, we would like to recall the relation between ω_j's and the harmonic measures. Let $\omega(z, F; \Omega)$ be the harmonic measure of the set $F \subset E$ in the domain $\Omega = \mathbb{C} \setminus E$ with respect to $z \in \Omega$. In $\overline{\mathbb{C}} \setminus E_J$ we have

$$\omega_k = \pi\omega(\infty, E_J^k), \ E_J^k = E_J \cap [\mathbf{b}_0, \mathbf{a}_k], \tag{4.12}$$

and

$$\omega_k = 2\pi\omega(\infty, E_T^k) = 2\pi\omega(0, E_T^k), \ E_T^k = E_T \cap \{e^{iz} : \ z \in [0, \mathbf{a}_k]\} \tag{4.13}$$

in $\overline{\mathbb{C}} \setminus E_T$.

Relations of the Martin's function in Ω with the complex Martin's function of the group Γ will be discussed in the next subsection.

4.2.2 Szegő kernel on the universal covering and Ahlfors problem

Recall that in this section $H^2(\alpha)$ denotes the Hardy space defined in Section 2.2.2. That is, we use $\mathbb{C}_+$ and not $\mathbb{D}$ as covering domain, but still consider the Hardy spaces corresponding to the integration with respect to the Lebesgue measure.

Lemma 4.11. *If $H^2(\alpha) \neq \{0\}$ for some α, then*

$$\sum_{\gamma\in\Gamma} \gamma'(\xi) < \infty \tag{4.14}$$

for almost all $\xi \in \mathbb{R}$.

Proof. Let $0 \neq g \in H^2(\alpha)$. Hence $f = |g|^2$ is a positive single-valued function and $f \in L^1$. Moreover, let $\mathbb{E} = \partial\mathcal{F} \cap \mathbb{R}$. It is a measurable fundamental set for the action of Γ on $\mathbb{R}$. Hence

$$\frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)d\xi = \frac{1}{2\pi}\int_{\mathbb{E}}\{\sum_{\gamma\in\Gamma}\gamma'(\xi)\}f(\xi)d\xi < \infty,$$

and we get (4.14). □

For a reason that will be clear in a moment we consider Γ as a subgroup of $SL_2(\mathbb{R})$ (see Remark 4.13). Recall that

$$SL(2,\mathbb{R}) = \left\{\gamma = \begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix} : \gamma_{ij} \in \mathbb{R}, \det\gamma = 1\right\},$$

acting on $\mathbb{C}_+$ by

$$\lambda \mapsto \frac{\gamma_{11}\lambda + \gamma_{12}}{\gamma_{21}\lambda + \gamma_{22}}.$$

One can readily check that

$$\operatorname{Im}(\gamma(\lambda)) = (\operatorname{Im}\lambda)|\gamma'(\lambda)|.$$

Thus, the condition (4.14) guarantees that the following series converges in the upper half-plane

$$\sum_{\gamma\in\Gamma}\operatorname{Im}\gamma(\lambda) = \sum_{\gamma\in\Gamma}\frac{\operatorname{Im}\lambda}{|\gamma_{21}\lambda+\gamma_{22}|^2}. \tag{4.15}$$

In other words the mass-point measure supported on the trajectory of infinity $\xi_\gamma = -\frac{\gamma_{22}}{\gamma_{21}} = \gamma^{-1}(\infty)$ with masses $\sigma_\gamma = \frac{1}{\gamma_{21}^2}$ satisfies the condition

$$\sum_{\gamma\in\Gamma,\gamma\neq 1_\Gamma}\frac{\sigma_\gamma}{1+\xi_\gamma^2} = \sum_{\gamma\in\Gamma,\gamma\neq 1_\Gamma}\frac{1}{\gamma_{21}^2+\gamma_{22}^2} < \infty.$$

Thus, the corresponding function

$$\mathcal{M}(\lambda) = \lambda + \sum_{\gamma\in\Gamma,\gamma\neq 1_\Gamma}\frac{1+\lambda\xi_\gamma}{\xi_\gamma-\lambda}\frac{\sigma_\gamma}{1+\xi_\gamma^2}$$

is well defined, has positive imaginary part given by (4.15), moreover, $\operatorname{Im}\mathcal{M}(\gamma(\lambda)) = \operatorname{Im}\mathcal{M}(\lambda)$. That is, $e^{i\mathcal{M}(\lambda)}$ is the complex Martin's function of the group Γ with respect to infinity, $\operatorname{Im}\mathcal{M}(\lambda) = M_\Omega(z(\lambda))$ and for $z(\lambda) \simeq \lambda^2$, $\lambda = i\eta$, $\eta \to \infty$, we have

$$\mathcal{M}(\lambda) - \mathcal{M}(0) = \theta_S(z(\lambda)) = -i\int_0^{z(\lambda)}\prod_{j=1}^{g}\frac{z-\mathbf{d}_j}{\sqrt{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}}\frac{dz}{2\sqrt{-z}}. \tag{4.16}$$

The Blaschke product along the trajectories corresponding to the critical points

$$\mathcal{W}(\lambda) = \prod_{j=1}^{g} b_{\mathbf{d}_j}(\lambda)$$

is called the Widom function. Note that convergence of this product is called Widom condition for the given group (domain). Such Fuchsian groups were studied by Pommerenke [51]. For a complement of a system of intervals the Widom condition holds obviously. Recall that a function is of bounded characteristic in the upper half-plane if it can be represented as a ratio of two bounded functions, cf. e.g. [20, 28].

Theorem 4.12 (Pommerenke). *The function $\mathcal{M}'(\lambda)$ is holomorphic in the upper half-plane with zeros at $\{z^{-1}(\mathbf{d}_j)\}_{j=1}^g$, see (4.16). Moreover, it is of bounded characteristic in the upper half-plane such that*

$$\frac{d\mathcal{M}(\lambda)}{d\lambda} = \sum_{\gamma\in\Gamma} \frac{1}{(\gamma_{21}\lambda+\gamma_{22})^2} = \sum_{\gamma\in\Gamma} \gamma'(\lambda) = \mathcal{W}(\lambda)\varpi(\lambda)^2, \tag{4.17}$$

where ϖ is an outer function. Respectively, for its boundary values on the real axis, one has

$$\frac{d\mathcal{M}(\lambda)}{d\lambda} = |\varpi(\lambda)|^2 = \sum_{\gamma\in\Gamma} \frac{1}{|\gamma_{21}\lambda+\gamma_{22}|^2} \geq 1, \quad \mathcal{W}(\lambda)\overline{\varpi(\lambda)} = \varpi(\lambda). \tag{4.18}$$

For an analytic function in $\mathbb{C}_+$, $\gamma\in\Gamma$, we write

$$f|[\gamma] = \frac{f(\gamma(\lambda))}{(\gamma_{21}\lambda+\gamma_{22})}, \quad \gamma = \begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix}. \tag{4.19}$$

Let $\alpha_{\mathcal{W}}\in\Gamma^*$ be the character of the Widom function $\mathcal{W}$. Then, see (4.17),

$$(\mathcal{M}(\gamma(\lambda)))' = \mathcal{M}'(\lambda) \Rightarrow \varpi|[\gamma] = \nu(\gamma)\varpi, \tag{4.20}$$

where ν denotes a certain fixed root of the character $\alpha_{\mathcal{W}}^{-1}$, $\nu^2 = \alpha_{\mathcal{W}}^{-1}$.

Remark 4.13. Generally, a square root of a character is defined up to a half-period, that is, up to a character $\mathfrak{j}\in\Gamma^*$ such that $\mathfrak{j}^2 = 1_{\Gamma^*}$. Note that in the definition (4.19) it is essential that Γ is considered as a subgroup of $SL_2(\mathbb{R})$, but not as a Fuchsian group, where $\pm\gamma$ generates the same transform. In fact, such passage is also defined up to a choice of a half-period. Indeed, any of the groups

$$\Gamma_{\mathfrak{j}} = \{\mathfrak{j}(\gamma)\gamma : \ \gamma\in\Gamma\} \subset SL_2(\mathbb{R}), \quad \mathfrak{j}\in\Gamma^*,\ \mathfrak{j}^2 = 1_{\Gamma^*}, \tag{4.21}$$

generates the same group of linear fractional transforms.

Recall that a function of bounded characteristic in $\mathbb{C}_+$ is of Smirnov class, or of Nevanlinna class $\mathcal{N}_+$ in another terminology [28, Chap. II Sect. 5], if it can be represented as a ratio of two functions from H^∞ with an *outer* denominator. Note that functions of this class obey a maximum principle of a high generality. For instance, if $f\in\mathcal{N}_+$ and its boundary values satisfy $f\in L^2$, then $f\in H^2$.

Definition 4.14. For $\alpha \in \Gamma^*$ the space $A_1^2(\Gamma, \alpha)$ is formed by those analytic functions f in $\mathbb{C}_+$ that satisfy the following three conditions

(i) f is of Smirnov class,

(ii) $f|[\gamma] = \alpha(\gamma) f \ \forall \gamma \in \Gamma$,

(iii) $\frac{1}{2\pi}\int_{\mathbb{E}} |f(\lambda)|^2 d\lambda < \infty$.

For our purpose these spaces are convenient, because of their nice description of the orthogonal complement of $A_1^2(\Gamma, \alpha)$, i.e., $L^2_{d\lambda|\mathbb{E}} \ominus A_1^2(\Gamma, \alpha) = \overline{A_1^2(\Gamma, \alpha^{-1})}$, see [71]. For an example, where these spaces appear naturally we refer to [49].

Proposition 4.15. *The following map $f \mapsto \varpi f$ sets a unitary correspondence between $H^2(\alpha)$ and $A_1^2(\Gamma, \nu\alpha)$.*

Proof. Let $f \in H^2(\alpha)$. Then

$$\|f\|^2 = \frac{1}{2\pi}\int_{\mathbb{E}} |f(\lambda)|^2 \sum_{\gamma\in\Gamma} \gamma'(\lambda) d\lambda = \frac{1}{2\pi}\int_{\mathbb{E}} |f(\lambda)\varpi(\lambda)|^2 d\lambda. \tag{4.22}$$

Since $f \in H^2$ and ϖ is outer $f\varpi \in \mathcal{N}_+$. The property (ii) follows from (4.20). Conversely, if $g \in A_1^2(\Gamma, \nu\alpha)$, then $f = g/\varpi$ is in the standard L^2 and of Smirnov class. Thus, it belongs to H^2. The ratio of two forms, see (ii), generates a function with character α. □

The point evaluation functional is bounded in H^2. By $k^\alpha_{\lambda_0}(\lambda) = k^\alpha(\lambda, \lambda_0)$ we denote the reproducing kernel in $H^2(\alpha)$, $\langle f, k^\alpha_{\lambda_0}\rangle = f(\lambda_0)$ for all $f \in H^2(\alpha)$.

As it was mentioned in Garabedian's original paper [27], the Ahlfors problem for single-valued functions is the easiest version of Nevanlinna-Pick kind problems for multiply connected domains (Riemann surfaces). They were studied later by Abrahamse [1] and in many consequent papers, see e.g. [10, 37]. The statement below is a consequence of Abrahamse's theorem. Note that a similar statement can be founded in an essentially much more general situation in [66].

Proposition 4.16.

$$\sup_{w\in H^\infty(\beta),\ \|w\|\le 1} |w(\lambda_0)|^2 = \inf_{\alpha\in\Gamma^*} \frac{k^{\alpha\beta}_{Sz}(\lambda_0,\lambda_0)}{k^{\alpha}_{Sz}(\lambda_0,\lambda_0)} = \frac{k^{\alpha_0\beta}_{Sz}(\lambda_0,\lambda_0)}{k^{\alpha_0}_{Sz}(\lambda_0,\lambda_0)}, \quad \alpha_0 = \alpha_0(\lambda_0). \tag{4.23}$$

Moreover, an extremal function $w_{\lambda_0,\beta}(\lambda)$ is a Blaschke product and

$$w_{\lambda_0,\beta}(\lambda)\overline{w_{\lambda_0,\beta}(\lambda_0)} = \frac{k^{\alpha_0\beta}_{Sz}(\lambda,\lambda_0)}{k^{\alpha_0}_{Sz}(\lambda,\lambda_0)}. \tag{4.24}$$

Note that one side of the statement deals with the trivial estimation

$$\begin{aligned} |w(\lambda_0)k^\alpha_{Sz}(\lambda_0,\lambda_0)|^2 =& |\langle w k^\alpha_{Sz,\lambda_0}, k^{\alpha\beta}_{Sz,\lambda_0}\rangle|^2 \\ \le& \|k^\alpha_{Sz,\lambda_0}\|^2 \|k^{\alpha\beta}_{Sz,\lambda_0}\|^2 = k^\alpha_{Sz,\lambda_0}(\lambda_0) k^{\alpha\beta}_{Sz,\lambda_0}(\lambda_0). \end{aligned} \tag{4.25}$$

Also, due to the nature of formulas (4.23), (4.24), it does not matter to use Szegő kernels, or the reproducing kernels of character automorphic Hardy spaces.

Proof of (2.23) *in Theorem 2.18.* Let v be an extremal function for Problem 7. Then $v = b_{z_0} w$, where $z_0 = z(\lambda_0)$ and $w = w_{\lambda_0, \beta\mu_{z_0}^{-1}}$ is an extremal function from (4.24).

Now we use duality [71] $L^2_{d\lambda|\mathbb{E}} \ominus A^2_1(\Gamma, \alpha) = \overline{A^2_1(\Gamma, \alpha^{-1})}$, $\forall \alpha \in \Gamma^*$, due to which

$$b_{z_0}(\lambda) \frac{\overline{k^{\alpha}_{Sz}(\lambda, \lambda_0)}}{\|k^{\alpha}_{Sz,\lambda_0}\|} = \frac{k^{\alpha^{-1}\mu_{z_0}}_{Sz}(\lambda, \lambda_0)}{\|k^{\alpha^{-1}\mu_{z_0}}_{Sz,\lambda_0}\|}, \quad \lambda \in \mathbb{R},$$

and

$$k^{\alpha^{-1}\mu_{z_0}}_{Sz}(\lambda_0, \lambda_0) k^{\alpha}_{Sz}(\lambda_0, \lambda_0) = |b'_{z_0}(\lambda_0)|^2. \tag{4.26}$$

Using $|w(\lambda)| = 1$ on $\mathbb{R}$ we get, simultaneously (4.24) and the dual representation

$$w(\lambda)\overline{w(\lambda_0)} = \frac{k^{\alpha_0\beta}_{Sz}(\lambda, \lambda_0)}{k^{\alpha_0\mu_{z_0}}_{Sz}(\lambda, \lambda_0)} = \frac{k^{\alpha_0^{-1}}_{Sz}(\lambda, \lambda_0)}{k^{\alpha_0^{-1}\beta^{-1}\mu_{z_0}}_{Sz}(\lambda, \lambda_0)}. \tag{4.27}$$

Generically, a solution of the problem is a unique Blaschke product of g complex Green's functions, see Theorem 4.17 and [1, 37], that is, $\alpha_0 = \alpha_0^{-1}\beta^{-1}$ and for this character

$$|v'(\lambda_0)|^2 = |b'_{z_0}(\lambda_0)|^2 |w(\lambda_0)|^2 = |b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha_0\beta}_{Sz}(\lambda_0, \lambda_0)}{k^{\alpha_0\mu_{z_0}}_{Sz}(\lambda_0, \lambda_0)}.$$

By (4.26), we have

$$|b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha^{-1}\beta}_{Sz}(\lambda_0, \lambda_0)}{k^{\alpha^{-1}\mu_{z_0}}_{Sz}(\lambda_0, \lambda_0)} = |b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha}_{Sz}(\lambda_0, \lambda_0)}{k^{\alpha^{-1}\mu_{z_0}}_{Sz}(\lambda_0, \lambda_0)} = k^{\alpha}_{Sz}(\lambda_0, \lambda_0)^2$$

for an arbitrary α such that $\alpha^2 = \beta$. Due to (4.23), we obtain (2.23) with $\alpha(\lambda_0) = \alpha_0^{-1}$. In degenerated cases the formula holds by continuity, although the choice for α_0 is not unique. □

Now we will essentially specify the second statement of Theorem 2.18 for finitely connected Denjoy domains. Note that the general formula for the analytic capacity in finitely-connected Denjoy domains is due to Pommerenke [50]. We presented this already in (2.8).

Theorem 4.17. *Let* $\Omega = \mathbb{C} \setminus E_S$ *and*

$$W(z) = \frac{1}{\sqrt{-z}} \prod_{j=1}^{g} \sqrt{\frac{z - \mathbf{a}_j}{z - \mathbf{b}_j}}. \tag{4.28}$$

Then

$$k_{Sz}(\lambda_0, \lambda_0) = \frac{\operatorname{Im} W(z_0)}{2 \operatorname{Im} z_0 |W(z_0)|} \left| \frac{dz}{d\lambda}(\lambda_0) \right|, \quad z_0 = z(\lambda_0). \tag{4.29}$$

Respectively, the extremal Ahlfors function for single-valued functions, i.e., for the trivial character, is of the form

$$w_{z_0,\Omega}(z) = \frac{z - z_0}{z - \overline{z}_0} \frac{W(z) - W(\overline{z}_0)}{W(z) + W(z_0)}. \tag{4.30}$$

First we prove the following lemma.

Lemma 4.18. *Let H^2_W be the space of Smirnov class functions $F(z)$ in Ω with the scalar product*

$$\|F\|^2_W = \frac{1}{\pi}\int_E \frac{|F(x+i0)|^2 + |F(x-i0)|^2}{2}\operatorname{Im} W(x)\mathrm{d}x = \frac{1}{2\pi i}\oint_E |F(x)|^2 W(x)\mathrm{d}x.$$

Then the reproducing kernel of this space is of the form

$$K_W(z,z_0) = \frac{-W(z)^{-1} + W(\overline{z}_0)^{-1}}{2(z-\overline{z}_0)}. \tag{4.31}$$

Proof. Note that $\overline{W(x)} = -W(x)$ if $x \in E$ and $\overline{W(\overline{z})} = W(z)$ for $z \in \Omega$. For $F \in H^2_W$ we can apply the Cauchy theorem

$$\langle F, K_W(\cdot, z_0)\rangle = \frac{1}{2\pi i}\oint_E \frac{W(x)^{-1} + W(z_0)^{-1}}{2(x-z_0)} F(x)W(x)\mathrm{d}x = F(z_0).$$

□

Proof of Theorem 4.17. According to Proposition 4.15 the space $A^2_1(\Gamma)$ can be interpreted as the collection of functions $f\varpi$, where $f \in H^2(\nu^{-1})$. In its turn, to $f \in H^2(\nu^{-1})$ we associate a multi-valued function F in Ω such that $F(z(\lambda)) = f(\lambda)$. Note that $|F(z)|$ is single-valued in Ω, and the scalar product, by (4.18) and (4.22), has the form

$$\|f\|^2_{H^2(\nu^{-1})} = \frac{1}{\pi}\int_E \frac{|F(x+i0)|^2 + |F(x-i0)|^2}{2}\prod_{j=1}^{g}\frac{x-\mathbf{d}_j}{\sqrt{(x-\mathbf{a}_j)(x-\mathbf{b}_j)}}\frac{\mathrm{d}x}{2\sqrt{x}}.$$

Let ϕ_W be the multi-valued outer function in Ω given by

$$\phi_W^{-2} = \frac{1}{2\mathcal{W}}\prod_{j=1}^{g}\frac{z-\mathbf{d}_j}{z-\mathbf{a}_j}. \tag{4.32}$$

Note that

$$|\phi_W|^{-2} = \frac{1}{2}\prod_{j=1}^{g}\frac{x-\mathbf{d}_j}{x-\mathbf{a}_j},\ x \in E,\ \text{and}\ \phi_W \circ \gamma = \nu^{-1}(\gamma)\phi_W$$

(at this point we fix the half-period for Γ, see Definition 4.19, so that the function ϕ_W and the form ϖ generate the mutually inverse characters). Since

$$\|f\|^2_{H^2(\nu^{-1})} = \frac{1}{\pi}\int_E \frac{|(F/\phi_W)(x+i0)|^2 + |(F/\phi_W)(x-i0)|^2}{2}\operatorname{Im} W(x)\mathrm{d}x,$$

for $g \in A^2_1(\Gamma)$ we obtain $g = G(z(\lambda))\phi_W\varpi$, where $G \in H^2_W$. Respectively,

$$k_{Sz}(\lambda,\lambda_0) = K_W(z(\lambda), z(\lambda_0))\phi_W(z(\lambda))\varpi(\lambda)\overline{\phi_W(z(\lambda_0))\varpi(\lambda_0)}. \tag{4.33}$$

By (4.17) and (4.32) we have

$$\varpi^2\phi_W^2 = -i\frac{1}{\mathcal{W}}\prod_{j=1}^{g}\frac{z-\mathbf{d}_j}{\sqrt{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}}\frac{1}{2\sqrt{-z}}\frac{\mathrm{d}z}{\mathrm{d}\lambda} \times 2\mathcal{W}\prod_{j=1}^{g}\frac{z-\mathbf{a}_j}{z-\mathbf{d}_j} = -iW(z)\frac{\mathrm{d}z}{\mathrm{d}\lambda}.$$

Thus (4.31) and (4.33) imply (4.29).

It remains to explain the extremal property of the fixed half-period. Consider the following family of functions

$$W_{\epsilon}(z) = W(z)\prod_{k=1}^{g}\left(\frac{z-\mathbf{b}_k}{z-\mathbf{a}_k}\right)^{\frac{1-\varepsilon_k}{2}},$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_g)$ and $\epsilon_k = \pm 1$. Similarly to Lemma 4.18, we have 2^g Hilbert spaces $H^2_{W_\epsilon}$ with the reproducing kernels

$$K_{W_\epsilon}(z, z_0) = \frac{-W_\epsilon(z)^{-1} + W_\epsilon(\overline{z}_0)^{-1}}{2(z-\overline{z}_0)}.$$

We define collection of outer functions

$$\psi_\epsilon^2 = \prod_{k=1}^{g}\left(\frac{z-\mathbf{b}_k}{z-\mathbf{a}_k}\right)^{\frac{1-\varepsilon_k}{2}}$$

whose characters (for the already fixed Γ) form all possible 2^g half-periods on this group. In this case

$$A^2_1(\Gamma, \mathfrak{j}) = \{g = G(z(\lambda))\varpi\phi_W\psi_\epsilon : \ G \in H^2_{W_\epsilon}\}.$$

Thus, we have to compare the value

$$k_{Sz}(\lambda_0, \lambda_0; \Gamma, \mathfrak{j}) = \frac{\operatorname{Im} W_\epsilon(z_0)}{2\operatorname{Im} z_0 |W_\epsilon(z_0)|}\left|\frac{\mathrm{d}z}{\mathrm{d}\lambda}(\lambda_0)\right|$$

for all half-periods $\mathfrak{j}$ to choose the minimal one. We use the exponential representation

$$W_\epsilon(z) = Ce^{\int_{\mathbb{R}}\frac{1+xz}{x-z}\chi_\epsilon(x)\frac{\mathrm{d}x}{1+x^2}}, \ C>0, \ \chi_\epsilon(x) := \frac{1}{\pi}\arg W_\epsilon(x),$$

due to which

$$\frac{\operatorname{Im} W_\epsilon(z)}{|W_\epsilon(z)|} = \sin(I_\epsilon(z)), \quad I_\epsilon(z) = \int_{\mathbb{R}}\frac{\operatorname{Im} z}{|x-z|^2}\chi_\epsilon(x)\mathrm{d}x.$$

The minimal value in the last expression corresponds to the minimum between two extreme values

$$\min_{\varepsilon_k=\pm1}\sin(I_\epsilon(z)) = \min\{\sin(I_+(z)), \sin(I_-(z))\}, \ I_\pm(z) := I_{\varepsilon_k=\pm1,\forall k}(z).$$

For these two we have

$$\sin(I_-(z)) - \sin(I_+(z)) = 2\sin\left\{\frac{1}{2}\int_{\cup_{j=1}^{g}(\mathbf{a}_j,\mathbf{b}_j)}\frac{\operatorname{Im} z\mathrm{d}x}{|x-z|^2}\right\}\cos\left\{\frac{1}{2}\int_{\mathbb{R}_+}\frac{\operatorname{Im} z\mathrm{d}x}{|x-z|^2}\right\}.$$

Due to $\int_{\mathbb{R}}\frac{\operatorname{Im} z\mathrm{d}x}{|x-z|^2} = \pi$, we get $\sin(I_+(z)) < \sin(I_-(z))$. That is, the configuration $\varepsilon_k = 1, \forall k$, corresponds to the global minimum.

Note that this extremal choice of ε_k corresponds to the exceptional case when

$$\operatorname{Re} W_{\epsilon}(z) \geq 0$$

in the upper half-plane and therefore for all $z \in \Omega$. Therefore, $W(z) - W(\overline{z}_0)$ has $g+1$, that is, the *maximal possible* number of zeros, $\{\overline{z}_k\}_{k=0}^g$, in Ω. Respectively,

$$\frac{W(z(\lambda)) - W(\overline{z}_0)}{W(z(\lambda)) + W(\overline{z}_0)} = \prod_{k=0}^{g} b_{\overline{z}_k}(\lambda)$$

and we obtain (4.30). □

Definition 4.19. In what follows the choice of the group $\Gamma \subset SL_2(\mathbb{R})$ is assumed to match with the extremal half-period in the Ahlfors problem, see Remark 4.13. That is, cf. (2.24),

$$\mathcal{A}(\lambda_0, 1_{\Gamma^*}) = k_{Sz}(\lambda_0, \lambda_0; \Gamma, 1_{\Gamma^*}) = k_{Sz}(\lambda_0, \lambda_0).$$

To describe reproducing kernels in the general case we introduce the following notations and definitions, see [46, Chap. IIIa], see also [59].

Definition 4.20. Let $\mathcal{R}$ denote the hyperelliptic Riemann surface

$$\mathcal{R} = \left\{ p = (z, W) : \ W^2 = -\frac{1}{z} \prod_{j=1}^{g} \frac{z - \mathbf{a}_j}{z - \mathbf{b}_j} \right\}$$

and $\overline{\mathcal{R}}$ be its compactification. The upper sheet means the collection of points $\{p = (z, W) : \ \operatorname{Re} W > 0\}$ and we identify it with the domain $\Omega = \mathbb{C} \setminus E_S$, where $W = W(z)$ is well defined. Thus, for a generic point on $\mathcal{R}$ we can write $(z, 1)$, $z \in \Omega$, having in mind a point on the upper sheet and $(z, -1)$, $z \in \Omega$, for a point on the lower sheet. Note that $(\mathbf{a}_j, \pm 1)$ (respectively $(\mathbf{b}_j, \pm 1)$) denotes the same point on $\overline{\mathcal{R}}$. Define the class of divisors

$$D(E) = \{(\mathbf{x}_j, \varepsilon_j) : \ \mathbf{x}_j \in [\mathbf{a}_j, \mathbf{b}_j], \ \varepsilon_j = \pm 1\}_{j=1}^g.$$

To the given D we associate a multi-valued function in Ω (a generalization of (4.32))

$$\phi_D(z(\lambda)) = \sqrt{\frac{1}{2} \prod_{j=1}^{g} \frac{(z(\lambda) - \mathbf{x}_j) b_{\mathbf{d}_j}(\lambda)}{(z(\lambda) - \mathbf{d}_j) b_{\mathbf{x}_j}(\lambda)}} \prod_{j=1}^{g} b_{\mathbf{x}_j}^{\frac{1+\varepsilon_j}{2}}(\lambda), \tag{4.34}$$

which can be extended on $\mathcal{R}$; cf. (1.38). In this case D corresponds to zeros of ϕ_D. Note that poles correspond to the divisor $\{(\mathbf{d}_j, -1)\}_{j=1}^g$. The character generated by this function is denoted by α_D.

Let $T(z) = z \prod_{j=1}^g (z - \mathbf{a}_j)(z - \mathbf{b}_j)$ and $U_D(z) = \prod_{j=1}^g (z - \mathbf{x}_j)$ so that

$$\frac{U_D(z)}{\sqrt{-T(z)}} = W(z) \prod_{j=1}^{g} \frac{z - \mathbf{x}_j}{z - \mathbf{a}_j}.$$

has positive imaginary part in the upper half-plane of the upper sheet. Let

$$m_\pm^D(z) = m_\pm(z) := \frac{-\sqrt{-T(z)} \pm V_D(z)}{U_D(z)}, \tag{4.35}$$

where the polynomial $V_D(z)$, $V_D(0) = 0$, of degree g is uniquely defined by the condition that on $\overline{\mathcal{R}}$ the function $m_+(z)$ has poles exactly at points forming the divisor D (and, by construction, at infinity). Let us point out that both functions have positive imaginary values in the upper half-plane, and $\overline{m_+(x)} = -m_-(x)$, $x \in E_S$. Respectively, $D_* := \{(\mathbf{x}_j, -\varepsilon_j)\}_{j=1}^g$ is the divisor, which generates $m_-(z)$, i.e., $m_-^D(z) = m_+^{D_*}(z)$.

Theorem 4.21. *For an arbitrary character $\alpha \in \Gamma^*$ there exists a unique divisor D such that $\alpha = \alpha_D$ with the character generated by ϕ_D (4.34). Let $H^2_{m_+^D}$ be the space of meromorphic functions $F(z)$ in Ω such that $F(z(\lambda))\phi_D(z(\lambda))$ is of Smirnov class equipped with the scalar product*

$$\|F\|^2_{m_+^D} = \frac{1}{\pi}\int_E \frac{|F(x+i0)|^2 + |F(x-i0)|^2}{2}\,\mathrm{Im}\,\frac{U_D(x)}{\sqrt{-T(x)}}\mathrm{d}x.$$

Then the reproducing kernel of this space is of the form

$$K_{m_+^D}(z, z_0) = \frac{m_+^D(z) - m_+^D(\overline{z}_0)}{2(z - \overline{z}_0)}, \quad z_0 \neq \mathbf{x}_k. \tag{4.36}$$

Consequently, the reproducing kernel of the space $A_1^2(\Gamma, \alpha\nu)$ is of the form

$$k_{Sz}(\lambda, \lambda_0; \Gamma, \alpha\nu) = K_{m_+^D}(z(\lambda), z(\lambda_0))\phi_D(z(\lambda))\varpi(\lambda)\overline{\phi_D(z(\lambda_0))\varpi(\lambda_0)}.$$

Proof of Theorem 4.21 and Theorem 2.18. Existence and uniqueness of D for the given α follows from Abel-Jacobi inversion theorem. Since poles of $m_\pm^D$ complement each other in Ω, see (4.35), we get by the Cauchy theorem

$$\langle F, K_{m_+^D}(\cdot, z_0)\rangle = \frac{1}{2\pi i}\oint_E \frac{-m_-^D(x) - m_+^D(z_0)}{2(x - z_0)} F(x)\frac{U_D(x)}{\sqrt{-T(x)}}dx = F(z_0).$$

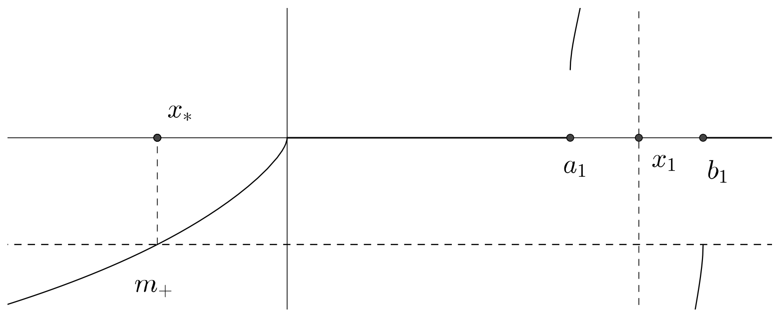


Figure 4.3: Change by a half-period is required if $z_0 \in (x_*, 0)$

It remains to prove the last statement of Theorem 2.18. Consider an elliptic case. By (4.27) we are interested in the reproducing kernel $k^\alpha_{Sz}(\lambda, \lambda_0)$ such that $\alpha^2 = \beta$ and the corresponding $K_{m_+^D}(z, z_0)$ has zero in the gap $(\mathbf{a}_1, \mathbf{b}_1)$ (on the upper sheet). We use the representation (4.36), assuming that $\mathbf{x}_1 \in (\mathbf{a}_1, \mathbf{b}_1)$ is a pole of m_+^D. Note that in this case $m_+^D(\mathbf{a}_1) > 0$ and $m_+^D(\mathbf{b}_1) < 0$. Therefore, see a sketch of a graph of $m_+^D(z)$ in Fig. 4.3, as soon as $z_0 < x_*$ the reproducing kernel $K_{m_+^D}(z, z_0)$ has indeed a zero in the gap $(\mathbf{a}_1, \mathbf{b}_1)$, but for $0 > z_0 > x_*$, $m_+^D(x) \neq m_+^D(z_0)$ for all $x \in [\mathbf{a}_1, \mathbf{b}_1]$ (the corresponding zero belongs to the lower sheet). Thus, in this range a change of the half-period is required (an extremal function corresponds to another divisor D).

□

4.2.3 Ahlfors problem and Abel-Jacobi inversion

Recall that Abelian differentials of the first kind on the Riemann surface $\mathcal{R}$, see Definition 4.20, form a g-dimensional linear space. We will fix a basis in this space such that the matrix of A-periods is the identity matrix, i.e.,

$$\mathrm{d}j_1^k = \frac{Q_k(z)}{\sqrt{-T(z)}}\mathrm{d}z,\ \deg Q_k = g-1, \quad \int_{\mathbf{a}_j}^{\mathbf{b}_j} \mathrm{d}j_1^k = \frac{1}{2}\delta_{k,j},\ k,j = 1,\dots,g, \quad (4.37)$$

where $\delta_{k,j}$ is the Kronecker symbol and integration is given along an interval on the upper sheet. Note that $2\operatorname{Re}\int_0^z \mathrm{d}j_1^k = \omega(z, E_k; \Omega)$ in Ω, as before $\omega(z, F; \Omega)$ denotes the harmonic measure in Ω, see (4.12), and $E_k = E_S \setminus [0, \mathbf{a}_k]$. Generally the Abel-Jacobi theorem sets a one-to-one correspondence between the Jacobian variety $\mathrm{Jac}(\mathcal{R})$ and the complex g-dimensional torus $\mathbb{C}^g/L_{\mathcal{R}}$, where $L_{\mathcal{R}}$ is the lattice generated by the matrix of periods of the Abelian integrals (4.37), see Theorem A.1 and Theorem A.2.

Now, we are interested just in the *real* part of the Abel-Jacobi inversion problem. Note that it plays a crucial role in the theory of finite gap self-adjoint/unitary operators, see e.g. [7, 9, 56]. Let us recall the Abel map from the preliminary section:

Theorem 4.22. *Let us fix a base point $D_0 = \{\mathbf{a}_j\}_{j=1}^g$ in the collection of divisors $D(E)$ on $\mathcal{R}$ and define the map from $D(E)$ to the standard real torus $\mathbb{R}^g/\mathbb{Z}^g$ by*

$$\alpha_k(D) = \sum_{j=1}^g \int_{\mathbf{a}_j}^{(\mathbf{x}_j,\varepsilon_j)} dj_1^k,\ \ D = \{(\mathbf{x}_j, \varepsilon_j)\}_{j=1}^g \in D(E),\ \ \alpha = \{\alpha_k\}_{k=1}^g \in \mathbb{R}^g/\mathbb{Z}^g, \quad (4.38)$$

with the integration along the interval on the lower or upper sheet depending on ε_j. This map is one-to-one.

Evidently, (4.38) can be rewritten as

$$\alpha_k = \sum_{j=1}^g \frac{\varepsilon_j}{2}\omega(\mathbf{x}_j, E_k) \quad \mod 1. \quad (4.39)$$

In the context of the Chebyshev problematics, we have a quite similar, but actually different inversion problem. For a fixed β and $\mathbf{x}_0 < 0$ the extremal function from Proposition 4.16 has the form of the Blaschke product

$$w_{\mathbf{x}_0,\beta}(\lambda) = \prod b_{\mathbf{x}_j}(\lambda), \quad (4.40)$$

and for its character we get

$$\beta_k = \sum_{j=1}^{g} \omega(\mathbf{x}_j, E_k) \mod 1. \tag{4.41}$$

Thus, the relation between β and the divisor $\{(\mathbf{x}_j, 1)\}_{j=1}^{g}$ can not be reduced exactly to the standard Abel-Jacobi inversion (Theorem 4.22). Indeed, directly from (4.41) we can only get

$$\alpha_k = \sum_{j=1}^{g} \frac{1}{2}\omega(\mathbf{x}_j, E_k) \mod 1,$$

where $2\alpha_k = \beta_k$ is still defined up to a half-period. If one of the solutions $\{\mathring{\alpha}_k\}$ is fixed, the whole collection is of the form $\{\mathring{\alpha}_k + \frac{1-\delta_k}{4}\}_{k=1}^{g}$, $\delta_k = \pm 1$. On this way to find $\mathbf{x}_j$'s, we have to solve (4.38) for all 2^g possible collections $\{\delta_k\}_{k=1}^{g}$ and after that to choose among all 2^g solutions the divisor $D = \{(\mathbf{x}_j, \varepsilon_j)\}_{j=1}^{g}$ with $\varepsilon_j = 1$, see (4.39).

Therefore, in our context Theorem 4.22 has to be substituted by the following claim.

Proposition 4.23. *Let I_j be the interval $[\mathbf{a}_j, \mathbf{b}_j]$ with the identification of the endpoints and equipped with the corresponding topology of the unit circle. Let $\mathcal{I} = \times_{j=1}^{g} I_j$ be the topological g-dimensional torus. Then the map $\mathcal{I} \to \mathbb{R}^g/\mathbb{Z}^g$ given by*

$$\beta_k(\mathfrak{X}) = \sum_{j=1}^{g} \omega(\mathbf{x}_j, E_k), \ \mathfrak{X} = \{\mathbf{x}_j\}_{j=1}^{g} \in \mathcal{I}, \ \beta = \{\beta_k\}_{k=1}^{g} \in \mathbb{R}^g/\mathbb{Z}^g, \tag{4.42}$$

is a homeomorphism.

Proof. We fix $\mathbf{x}_0$ in the complementary gap and for the given β get an extremal Blaschke product $w_{\mathbf{x}_0,\beta}$ in the form (4.40). Due to (4.41) we have existence in (4.42).

To get uniqueness, we note that any Blaschke product $w_\beta(\lambda)$ with the character β, of the form (4.40) represents *an inner part* of a reproducing kernel $k_{Sz}^{\alpha_0\beta}(\lambda, \lambda_0)$, $\mathbf{x}_0 = z(\lambda_0)$, for a certain $\alpha_0 \in \Gamma^*$. (Note in brackets, that the position of *zeros of reproducing kernels* deals precisely with the Abel-Jacobi inversion (4.38), see Theorem 4.21.) Since an outer part of a reproducing kernel is also a reproducing kernel we get

$$k_{Sz}^{\alpha_0\beta}(\lambda, \lambda_0) = w_\beta(\lambda)\overline{w_\beta(\lambda_0)}k_{Sz}^{\alpha_0}(\lambda, \lambda_0).$$

Thus, see (4.25),

$$|w_\beta(\lambda_0)|^2 = \inf_{\alpha\in\Gamma^*} \frac{k_{Sz}^{\alpha\beta}(\lambda_0, \lambda_0)}{k_{Sz}^{\alpha}(\lambda_0, \lambda_0)} = \frac{k_{Sz}^{\alpha_0\beta}(\lambda_0, \lambda_0)}{k_{Sz}^{\alpha_0}(\lambda_0, \lambda_0)}.$$

Due to the uniqueness of the extremal function (in the normalization $w_\beta(\lambda_0) > 0$) we get uniqueness in (4.42) (up to an identification of the gap endpoints). □

4.2.4 Asymptotics: real case

Definition 4.24. We say that the comb Π_J (Π_S) is n-regular (ℓ-regular) if $\frac{n}{\pi}\omega_k \in \mathbb{Z}$ for all k (respectively, $\frac{\ell}{\pi}\omega_k \in \mathbb{Z}$).

We have already demonstrated in the preliminary section how Markov corrections method reveals the structure of Chebyshev polynomials associated to subsets of the real line. These method can also be applied to our setting.

Theorem 4.25. *Let $\mathbf{x} \in \mathbb{R}\setminus E_J$ ($\mathbf{x} \in \mathbb{R}\setminus E_S$). For a given n (ℓ) there exists an n-regular (ℓ-regular) comb $\tilde\Pi_{J,n,\mathbf{x}}$ ($\tilde\Pi_{S,\ell,\mathbf{x}}$) such that the extremal function of Problem 4 (Problem 6) is given in terms of the corresponding conformal mapping*

$$P_{n,\mathbf{x}}(z) = \cos n\tilde\theta_{J,n,\mathbf{x}}(z) \quad (F_{\ell,\mathbf{x}}(z) = \cos \ell\tilde\theta_{S,\ell,\mathbf{x}}(z)) \tag{4.43}$$

and

$$A_n(\mathbf{x}, E_J) = |P'_{n,\mathbf{x}}(\mathbf{x})| \quad (A_\ell(\mathbf{x}, E_S) = |F'_{\ell,\mathbf{x}}(\mathbf{x})|).$$

In this case, the set $\tilde E_{J,n,\mathbf{x}}$ ($\tilde E_{S,\ell,\mathbf{x}}$), which corresponds to the base of the regular comb, contains the initial set E_J (E_S) and, on the other hand, the original set contains the preimage of at least one of possibly two different points of the form $\frac{\pi k}{n}\pm 0$ ($\frac{\pi k}{\ell}\pm 0$) on the base of the regular comb for all $k = 0,\dots,n$ ($k\in\mathbb{Z}_+$).

We will prove a counterpart of this theorem related to Problem 5. For the Pell equation approach see [42]. A comb Π_T is called n-regular if $\frac{n}{2\pi}\omega_j \in \mathbb{Z}$ for all j, see Fig. 4.1.

Theorem 4.26. *Let $e^{i\mathbf{x}} \in \mathbb{T}\setminus E_T$. For a given n there exists an n-regular comb $\tilde\Pi_{T,n,e^{i\mathbf{x}}}$ such that the extremal function of Problem 5 is given in terms of the corresponding conformal mapping by*

$$P_{n,e^{i\mathbf{x}}}(e^{iz}) = e^{iz\frac{n}{2}}\cos\frac{n}{2}\tilde\theta_{T,n,e^{i\mathbf{x}}}(z) \quad \text{and} \quad A_n(e^{i\mathbf{x}}, E_T) = |P'_{n,e^{i\mathbf{x}}}(e^{i\mathbf{x}})|.$$

In this case, the set $\tilde E_{T,n,e^{i\mathbf{x}}}$, which corresponds to the base of the regular comb, contains the initial set E_T and, on the other hand, at least one of possibly two different points $e^{i\tilde\theta^{-1}_{T,n,e^{i\mathbf{x}}}(\frac{2\pi k}{n}\pm 0)}$ belongs to E_T for all $k = 0,\dots,n-1$.

Proof. WLOG we assume that $\mathbf{x} = \mathbf{x}_0 \in (\mathbf{a}_0, \mathbf{b}_0)$. Let $P(\zeta) = P_{n,e^{i\mathbf{x}_0}}(\zeta)$ be an extremal polynomial. We represent it in the form

$$P(e^{iz}) = e^{iz\frac{n}{2}}F(z). \tag{4.44}$$

Clearly $F(z)$ is a periodic entire function, $F(z+2\pi) = (-1)^n F(z)$, and $F(\mathbf{x}_0) = 0$. Since an extremal polynomial is given up to multiplication by a unimodular constant, we can assume that $F'(\mathbf{x}_0) > 0$. After that we can substitute $F(z)$ by its symmetric part $\frac{1}{2}(F(z)+\overline{F(\bar z)})$ and we still get an extremal polynomial $P(e^{iz})$. Thus, we can assume that $F(z)$ is real on the real axis in the representation (4.44).

We claim that $F(z)$ has no complex zeros, or, equivalently $P(\zeta)$ does not have zeros in $\mathbb{C}\setminus\mathbb{T}$. Indeed, let $F(z_0) = 0$, $\operatorname{Im} z_0 > 0$. Note that $F(\bar z_0) = 0$. Consider

$$Q(\zeta) = P(\zeta)\left(1 - \delta\frac{(\zeta - e^{i\mathbf{x}_0})(1-\zeta e^{-i\mathbf{x}_0})}{(\zeta - e^{iz_0})(1-\zeta e^{-i\bar z_0})}\right),\ \delta > 0.$$

Note that $Q(\zeta)$ is a polynomial of degree n such that

$$Q(e^{i\mathbf{x}_0}) = 0 \quad \text{and} \quad Q'(e^{i\mathbf{x}_0}) = P'(e^{i\mathbf{x}_0}). \tag{4.45}$$

In the same time, for $\zeta \in \mathbb{T}$

$$\frac{(\zeta - e^{i\mathbf{x}_0})(1 - \zeta e^{-i\mathbf{x}_0})}{(\zeta - e^{iz_0})(1 - \zeta e^{-i\bar{z}_0})} = \frac{|\zeta - e^{i\mathbf{x}_0}|^2}{|\zeta - e^{iz_0}|^2}.$$

That is, for a sufficiently small but positive δ we have

$$\max_{\zeta \in E_T} |P(\zeta)| > \max_{\zeta \in E_T} |Q(\zeta)|. \tag{4.46}$$

Thus $P(\zeta)$ was not an extremal polynomial.

In a similar way we prove that all zeros of $F(z)$ are simple. Now we prove that between two (necessary real) consequent zeros of this function, say z_1, z_2, there is a point $y \in (z_1, z_2)$ such that $|F(y)| = 1$ and $e^{iy} \in E_T$. Assuming that on the contrary $\{e^{iy}; y \in (z_1, z_2)\} \cap E_T = \emptyset$, or that $\max_{y \in (z_1,z_2)\cap E_T} |F(y)| < 1$, we define the polynomial

$$Q(\zeta) = e^{i\frac{n}{2}z} G(z), \quad G(z) = F(z)\left(1 - \delta \frac{\sin^2 \frac{z - \mathbf{x}_0}{2}}{\sin \frac{z - z_1}{2} \sin \frac{z - z_2}{2}}\right).$$

On the period we have to consider three regions: I_1 is a union of small vicinities of points z_1 and z_2; the interval $I_2 = (z_1 + \varepsilon, z_2 - \varepsilon)$; and the remaining set I_3. On I_1, $|G(z)|$ is strictly less than one if δ is small. In I_2 the factor in brackets is greater than one, but there is no restriction on $|G(y)|$ if no one of the points e^{iy}, $y \in I_2$, belongs to E_T. In the second case, $\max_{z \in I_2} |F(z)|$ is a fixed value, which is less than one. So, a small $\delta > 0$ can be chosen such that the product $|G(z)|$ is still less than one. On the remaining part I_3, $\max |G(z)| < 1$ due to the chosen correction factor. Since (4.45) and (4.46) hold, we get a contradiction. Note that the case $z_1 = \mathbf{x}_0$ ($z_2 = \mathbf{x}_0$) requires special, but basically the same consideration.

We can refer to general theorems [43, 58], or, having in mind periodicity of $F(z)$, just to count the number of ± 1 points (including multiplicity) on a period to conclude that all such points are real. Thus,

$$\tilde{\theta}_T(z) = \frac{2}{n} \arccos F(z), \quad \tilde{\theta}_T(\mathbf{x}_0) = \frac{\pi}{n},$$

is well defined in the upper half-plane. Making inspection of the boundary behaviour we conclude that this is a conformal mapping on a suitable comb $\tilde{\Pi}_T$. Moreover, this comb is n-regular, according to our definition.

Since $|F(z)| \le 1$ for $e^{iz} \in E_T$ and generally $|F(z)| \le 1$ if and only if $\tilde{\theta}_T(z) \in \mathbb{R}$, we obtain that

$$E_T \subset \tilde{E}_T = \{e^{iz} : \tilde{\theta}_T(z) \in \mathbb{R}\}.$$

Zeros of $F(z)$ correspond to $z_k : \tilde{\theta}_T(z_k) = \frac{\pi + 2\pi k}{n}$, $k \in \mathbb{Z}$. Therefore, between each consequent pair (z_{k-1}, z_k) there is a point $e^{iy} \in E_T$ such that $F(y) = \pm 1$. If the boundary of the domain $\tilde{\Pi}_T$ contains a slit with the base at $\tilde{\omega}_k = \frac{2\pi k}{n}$ then y corresponds either to the left or right limit point. Otherwise, this y corresponds to a single point $\tilde{\omega}_k$ on the boundary of $\tilde{\Pi}_T$.

Conversely, if we have an n-regular comb $\tilde{\Pi}_T$ and $\tilde{\theta}_T(z)$ is the comb function with the normalization $\tilde{\theta}_T(x_0) = \frac{\pi}{n}$, then

$$F(z) = \cos \frac{n}{2}\tilde{\theta}_T(z)$$

is an analytic function in the upper half-plane, and real-valued on the real axis. Being extended by the symmetry principle in the lower half-plane, it represents an entire function of exponential type $\frac{n}{2}$. Thus, $P(\zeta)$ of the form (4.44) is a polynomial of degree n. Every set, which contains one of possibly two different points $e^{i\tilde{\theta}^{-1}(\frac{2\pi k}{n}\pm 0)}$ for all $k = 0, \dots, n-1$, forms the so-called maximal Chebyshev set of alternation, see e.g. [58]. Due to the Chebyshev theorem $P(\zeta)$ is an extremal polynomial on an arbitrary E_T containing the given set of alternation. □

In the following remark, we present an analog to Proposition 1.4.

Remark 4.27. The set $\tilde{E}_T = \tilde{E}_{T,n,e^{ix_0}}$ represents an extension of the set E_T,

$$\tilde{E}_T = E_T \cup \{e^{iz};\ z \in \cup_{j=0}^n [\mathbf{u}_j, \mathbf{v}_j]\}, \quad [\mathbf{u}_j, \mathbf{v}_j] \subset [\mathbf{a}_j, \mathbf{b}_j].$$

A simple analysis shows that there are the following three possibilities of a proper extension in the gap $(\mathbf{a}_j, \mathbf{b}_j)$: for a suitable $k_j \in \mathbb{Z}$

(i) the gap is completely closed $\mathbf{u}_j = \mathbf{a}_j, \mathbf{v}_j = \mathbf{b}_j$:

$$\frac{2\pi k_j}{n} + 0 \le \tilde{\theta}_T(\mathbf{a}_j) < \tilde{\theta}_T(\mathbf{b}_j) \le \frac{2\pi(k_j+1)}{n} - 0.$$

(ii) a one-sided extension, say, $\mathbf{u}_j = \mathbf{a}_j$, $\mathbf{v}_j < \mathbf{b}_j$: there is a slit, $\tilde{h}_{k_j+1} > 0$, and

$$\tilde{\theta}_T(\mathbf{b}_j) = \frac{2\pi(k_j+1)}{n} + 0 \left(\tilde{\theta}_T(\mathbf{v}_j) = \frac{2\pi(k_j+1)}{n} - 0\right) \text{ and } \tilde{\theta}_T(\mathbf{a}_j) \ge \frac{2\pi k_j}{n} + 0;$$

(iii) $[\mathbf{u}_j, \mathbf{v}_j]$ is an internal subinterval: there are two slits, $\tilde{h}_{k_j} > 0$, $\tilde{h}_{k_j+1} > 0$, and

$$\tilde{\theta}_T(\mathbf{a}_j) = \frac{2\pi k_j}{n} - 0, \quad \tilde{\theta}_T(\mathbf{b}_j) = \frac{2\pi(k_j+1)}{n} + 0;$$

Due to (4.13) the harmonic measure in the origin of each additional arc $\{e^{iz} : z \in [\mathbf{u}_j, \mathbf{v}_j]\}$ in $\tilde{\Omega} = \overline{\mathbb{C}} \setminus \tilde{E}_T$ is not more than $\frac{1}{n}$. That is, the length of each additional arc tends to zero as $n \to \infty$. Thus, (E_T is fixed and $\mathbf{x}_0 \in (\mathbf{a}_0, \mathbf{b}_0)$) for sufficiently big n the case (i) is not possible, and we always have case (iii) for the chosen gap $(\mathbf{a}_0, \mathbf{b}_0)$, since $\mathbf{x}_0 \in [\mathbf{u}_0, \mathbf{v}_0]$.

Now we can pass to the limit in n.

Proof of Theorem 2.19. We give a proof for entire functions, other cases are quite similar. Let $F_{\ell,\mathbf{x}_0}(z)$, $\ell > 0$, $\mathbf{x}_0 < 0$, be the extremal function, see Theorem 4.25. Using compactness of Γ^* we choose a convergent sequence

$$\beta = \beta(\{\ell_k\}) = \lim_{k\to\infty} \alpha_{S,\ell_k}. \tag{4.47}$$

Using compactness of the family

$$\{F_{\ell,\mathbf{x}_0}(z(\lambda))e^{i\ell\mathcal{M}(\lambda)}\}_{\ell>0},$$

we choose an arbitrary subsequence of $\{\ell_k\}$ (but keeping the same notation) so that the following limit exist

$$w(\lambda) = w(\lambda; \{\ell_k\}) = \lim_{k\to\infty} F_{\ell_k,\mathbf{x}_0}(z(\lambda))e^{i\ell_k\mathcal{M}(\lambda)}.eadily \tag{4.48}$$

Now, consider the Martin's function $\tilde{M}(z) = \tilde{M}_{\ell,\mathbf{x}_0}(z) = \operatorname{Im}\tilde{\theta}_{S,\ell,\mathbf{x}_0}$ of the domain $\mathbb{C}\setminus\tilde{E}_{S,\ell,\mathbf{x}_0}$ in the original bigger domain Ω and solve the corresponding Poisson problem. Since $E_S\subset\tilde{E}_{S,\ell,\mathbf{x}_0}$ it vanishes on the boundary. Moreover, it is harmonic in the complement to the additional intervals $\cup_{j=0}^g[\mathbf{u}_j,\mathbf{v}_j] = \tilde{E}_{S,\ell,\mathbf{x}_0}\setminus E_S$. It is continuous in the whole Ω, but its normal derivative has jumps on the union of these intervals. Recall that $\tilde{\theta}(x) = \tilde{\theta}_{S,\ell,\mathbf{x}_0}(x)$ is real valued on $\tilde{E}_{S,\ell,\mathbf{x}_0}$. By the Cauchy-Riemann equations we have

$$\frac{\partial\tilde{M}}{\partial y}(x) = \frac{\partial\tilde{\theta}(x)}{\partial x}, \quad x\in\tilde{E}_{S,\ell,\mathbf{x}_0}\setminus E_S.$$

We note, that since $\tilde{\theta}$ is real on the additional intervals, $\mathrm{d}\tilde{\theta}$ is the generalized Laplacian of $\tilde{M}$. Therefore, the Poisson-Jensen formula [53, Theorem 4.5.1] gives

$$\tilde{M}_{S,\ell,\mathbf{x}_0}(z) = M_\Omega(z) - \frac{1}{\pi}\int_{\tilde{E}_{S,\ell,\mathbf{x}_0}\setminus E} G_\Omega(z,x)d\tilde{\theta}_{S,\ell,\mathbf{x}_0}(x). \tag{4.49}$$

Hence, we can apply the same trick as in the proof of Lemma 4.4. According to Remark 4.27, for a sufficiently large ℓ, each additional interval is of the form (ii) or (iii) and we have

$$\tilde{\theta}_{S,\ell,\mathbf{x}_0}(\mathbf{u}_j) - \tilde{\theta}_{S,\ell,\mathbf{x}_0}(\mathbf{v}_j)\begin{cases}\le\frac{\pi}{\ell}, & \text{case (ii)},\\ =\frac{\pi}{\ell}, & \text{case (iii)}.\end{cases} \tag{4.50}$$

Also recall that as $\ell\to\infty$ the ℓ-depending system of intervals $[\mathbf{u}_0^{(\ell)},\mathbf{v}_0^{(\ell)}]$ shrinks to the point $\mathbf{x}_0$. We again choose a subsequence, keeping the same notations, so that

$$\lim_{m\to\infty}\mathbf{u}_j^{(\ell_k)} = \lim_{m\to\infty}\mathbf{v}_j^{(\ell_k)} = \mathbf{x}_j, \quad j=1,\dots,g,$$

for some $\mathbf{x}_j\in[\mathbf{a}_j,\mathbf{b}_j]$. Since $G_\Omega(z,x)$ is continuous in x and $G_\Omega(z,\mathbf{a}_j) = G_\Omega(z,\mathbf{b}_j) = 0$, by (4.49) and (4.50), we obtain

$$\lim_{m\to\infty}\ell_k(\tilde{M}_{S,\ell_k,\mathbf{x}_0}(z) - M_\Omega(z)) = -\sum_{j=0}^g G_\Omega(z,\mathbf{x}_j). \tag{4.51}$$

Now we go back to (4.48). For $z=z(\lambda)$, by (4.43), we have

$$|w(\lambda)| = \lim_{k\to\infty} e^{\ell_k(\tilde{M}_{\ell_k,\mathbf{x}_0}(z)-M_\Omega(z))}\frac{\left|1+e^{2i\ell_k\tilde{\mathcal{M}}_{\ell_k,\mathbf{x}_0}(\lambda)}\right|}{2} = \frac{1}{2}\prod_{j=0}^g|b_{x_j}(\lambda)|,$$

and $\beta = \prod_{j=0}^{g} \mu_{\mathbf{x}_j}$. Since $\{\mathbf{x}_j\}$ with a suitable identifications, see Proposition 4.23, corresponds to the extremal function of Problem 7 for the given β, we conclude that

$$\lim_{k\to\infty} e^{-\ell_k M(z_0)} \left|\frac{dz}{d\lambda}(\lambda_0)\right| A_{\ell_k}(z_0, E_S) = \frac{1}{2}\mathcal{A}(\lambda_0, \beta)$$

along the *original* sequence $\{\ell_k\}$. Since β is an arbitrary character of the form (4.47), we get (2.27).

□

4.2.5 To make it complex

To complement proofs given in the previous section, we will discuss extremal polynomials $P_n(z) = P_{n,z_0}(z)$ of Problem 4. Obviously, $e^{ic}P_n(z)$, $c \in \mathbb{R}$, is also an extremal polynomial. So, a dual setting of the problem is the following: we fix an arbitrary non zero value of the derivative at z_0 and look for a polynomial $\tilde{P}_n(z)$ with the smallest maximum norm $\|\tilde{P}_n\|$ on E_J. In this case, $P_n(z) = \frac{\tilde{P}_n(z)}{\|\tilde{P}_n\|}$. Thus, due to the Kolmogorov criterion, see Theorem 1.12, $P_n(z)$ is extremal if and only if

$$\inf_{x\in E_J:\ |P_n(x)|=1} \operatorname{Re}(x-\overline{z}_0)^2 P_n(x)\overline{Q_{n-2}(x)} \le 0 \tag{4.52}$$

for an arbitrary polynomial Q_{n-2} of degree $n-2$.

Let

$$\frac{z-\overline{z}_0}{z-z_0}P_n(z) = \Phi_n(z) + i\Psi_n(z), \quad Q_{n-2}(z) = \mathcal{X}_{n-2}(z) + i\mathcal{Y}_{n-2}(z)$$

be the decompositions of the corresponding polynomials into the real and imaginary parts. Then (4.52) is of the form

$$\inf_{x\in E_J:\ \Phi_n(x)^2+\Psi_n(x)^2=1} \{\Phi_n(x)\mathcal{X}_{n-2}(x) + \Psi_n(x)\mathcal{Y}_{n-2}(x)\} \le 0.$$

Due to the symmetry properties it is enough to solve Problem 4 for z_0 in the upper half-plane $\mathbb{C}_+$. Evidently in this case $P_n(z)$ has all zeros, except for z_0, in the lower half-plane and

$$\left|\frac{i + \frac{\Psi_n(z)}{\Phi_n(z)}}{i - \frac{\Psi_n(z)}{\Phi_n(z)}}\right| \le 1, \quad z \in \mathbb{C}_+.$$

In other words $-\frac{\Psi_n(z)}{\Phi_n(z)}$ has positive imaginary part in the upper half-plane. This property implies that $-\frac{\Psi_n(z)}{\Phi_n(z)}$ is an increasing function between two consecutive poles. Therefore, the zeros of these two polynomials interlace.

Lemma 4.28. *Assume that for two real polynomials with $-\operatorname{Im}\frac{\Psi_n(z)}{\Phi_n(z)} \ge 0$, $z \in \mathbb{C}_+$, the set $X = \{x \in E_J :\ \Phi_n(x)^2 + \Psi_n(x)^2 = 1\}$ coincides with the collection of all zeros of $\Phi_n(x)$. If*

$$\Phi_n(x)^2 + \Psi_n(x)^2 \le 1,\ x \in E_J,$$

then

$$P_n(z) = \frac{z-z_0}{z-\overline{z}_0}(\Phi_n(z) + i\Psi_n(z))$$

is an extremal polynomial of Problem 4 for an arbitrary zero $\overline{z}_0$ of the complex polynomial $\Phi_n(z) + i\Psi_n(z)$.

Proof. In this case, due to the Kolmogorov criterion we have to check that

$$\inf_{x\in X} \Psi_n(x)\mathcal{Y}(x) \le 0$$

for an arbitrary real polynomial $\mathcal{Y}(x)$, $\deg \mathcal{Y} = n-2$. Assume that there exists $\mathcal{Y}(x)$, which violates this property. Since zeros of $\Phi_n(z)$ and $\Psi_n(z)$ interlace, and $\Psi_n(x)\mathcal{Y}(x) > 0$ for all $x \in X$, the polynomial $\mathcal{Y}(z)$ has $n-1$ zeros. That is, $\mathcal{Y}(z)$ is zero identically, and we get a contradiction. □

Let $\tilde{E}_{J,n}$ be an n-regular extension for the given set E_J and $\tilde{\theta}_n$ be the corresponding comb-function. We define associated polynomials Φ_n and Ψ_n by

$$e^{-in\tilde{\theta}_n(z)} = \cos n\tilde{\theta}_n(z) - i\sin n\tilde{\theta}_n(z) = \Psi_n(z) + \sqrt{\prod_{j=0}^{g}\frac{(z-\mathbf{u}_j)(z-\mathbf{v}_j)}{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}}\Phi_n(z), \tag{4.53}$$

and note that their zeros interlace. Let us point out that (4.53) is basically (1.3). Indeed, since sine is an odd function, we see as in the proof of Theorem 1.9 that

$$\Phi_n(z) = -i\sqrt{\prod_{j=0}^{g}\frac{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}{(z-\mathbf{u}_j)(z-\mathbf{v}_j)}}\sin n\tilde{\theta}_n(z)$$

is a real polynomial. Hence, by this choice, all double zeros of the Pell equation (1.2) (i.e. all base points $\tilde{\omega}_k = \frac{k\pi}{n}$ where $\tilde{h}_k = 0$) correspond to zeros of Φ_n.

Theorem 4.29. *Assume that for the given extension $\tilde{E}_{J,n}$*

$$\tilde{\rho}_n^2 := -\sup_{x\in E_J}\prod_{j=0}^{g}\frac{(x-\mathbf{u}_j)(x-\mathbf{v}_j)}{(x-\mathbf{a}_j)(x-\mathbf{b}_j)} > 0. \tag{4.54}$$

Let $\mathcal{Z}_n(\rho) = \{z_j\}_{j=0}^{g}$ be the collection of points conjugated to the zeros of

$$\mathfrak{p}_n(z,\rho) = \rho\Phi_n(z) + i\Psi_n(z), \quad \rho^2 < \tilde{\rho}_n^2. \tag{4.55}$$

Then

$$P_{n,z_j}(z) = \frac{z-z_j}{z-\overline{z}_j}\mathfrak{p}_n(z,\rho)$$

is the Ahlfors polynomial with respect to $z_j \in \mathcal{Z}_n(\rho)$ for the given set E_J.

Remark 4.30. To keep all zeros of $\mathfrak{p}_n(z,\rho)$ defined in (4.55) in the lower half-plane one has to choose $\rho < 0$ if $u_0 > \mathbf{a}_0$ and $\rho > 0$ if $v_0 < \mathbf{b}_0$ (note that the leading coefficient of $\Phi_n(z)$ is positive, due to the standard choice of the square root at infinity in (4.53)).

Proof of Theorem 4.29. We have

$$\Psi_n(x)^2 + \rho^2\Phi_n(x)^2 \le \Psi_n(x)^2 - \prod_{j=0}^{g}\frac{(x-\mathbf{u}_j)(x-\mathbf{v}_j)}{(x-\mathbf{a}_j)(x-\mathbf{b}_j)}\Phi_n(x)^2 = 1$$

for $x \in E_J$. Therefore we can use Lemma 4.28.

□

Lemma 4.31. *Assume that along a subsequence*

$$\lim_{k\to\infty}\mathbf{u}_m^{(n_k)}=\lim_{k\to\infty}\mathbf{v}_m^{(n_k)}=\mathbf{x}_m\in(\mathbf{a}_m,\mathbf{b}_m),\quad m=0,\dots,g. \tag{4.56}$$

Let $\mathfrak{X}=\{\mathbf{x}_m\}_{m=0}^g$ *and*

$$\tilde\rho^2=\tilde\rho^2(\mathfrak{X})=-\sup_{x\in E_J}\frac{U_{\mathfrak{X}}^2(x)}{R_E(x)}>0, \tag{4.57}$$

where

$$R_E(z)=\prod_{j=0}^g(z-\mathbf{a}_j)(z-\mathbf{b}_j),\quad U_{\mathfrak{X}}(z)=\prod_{j=0}^g(z-\mathbf{x}_j).$$

Then for $\rho^2<\tilde\rho^2$, $\rho(\mathbf{b}_0-\mathbf{x}_0)>0$, *and* $z=z(\lambda)$ *we have*

$$\lim_{k\to\infty}e^{in_k\theta_J(z)}\mathfrak{p}_{n_k}(z,\rho)=\frac{1}{2}\prod_{j=0}^g b_{\mathbf{x}_j}(\lambda)\left(\rho\frac{\sqrt{R_E(z)}}{U_{\mathfrak{X}}(z)}+i\right). \tag{4.58}$$

Proof. We use a counterpart of (4.51), $z=z(\lambda)$,

$$\lim_{n\to\infty}e^{in_k(\theta_J(z)-\tilde\theta_{J,n_k}(z))}=\prod_{j=0}^g b_{\mathbf{x}_j}(\lambda). \tag{4.59}$$

Therefore, due to the definition (4.53),

$$\lim_{k\to\infty}e^{in_k\theta(z)}\mathfrak{p}_{n_k}(z,\rho)=\lim_{k\to\infty}\left(i\frac{e^{in_k(\theta_J(z)-\theta_{J,n_k}(z))}+e^{in_k(\theta_J(z)+\theta_{J,n_k}(z))}}{2}\right.$$
$$\left.+\rho\sqrt{\prod_{j=0}^g\frac{(z-\mathbf{a}_j)(z-\mathbf{b}_j)}{(z-\mathbf{u}_j^{(n_k)})(z-\mathbf{v}_j^{(n_k)})}}\frac{e^{in_k(\theta_J(z)-\theta_{J,n_k}(z))}-e^{in_k(\theta_J(z)+\theta_{J,n_k}(z))}}{2}\right).$$

By (4.56) and (4.59) we get (4.58). □

Recall that any character is uniquely defined by its values on a system of free generators, $\beta_j=\beta(\gamma_j)$, $j=1,\dots,g$, see Fig. 4.2. Thus, $\beta=\prod_{j=0}^g\mu_{\mathbf{x}_j}$ is equivalent to

$$\sum_{j=0}^g\omega(\mathbf{x}_j,E_k)=\beta_k$$

where $E_k=E_J^k$, see (4.12), and $\omega(z,E_k)$ is the harmonic measure of E_k in Ω with respect to $z\in\Omega$, and according to Proposition 4.23 the set $\mathcal{I}$ and $\Gamma^*\simeq\mathbb{R}^g/\mathbb{Z}^g$ are homeomorphic.

Proof of Theorem 2.20. We follow to the line of the proof of Theorem 2.19. According to Lemma 4.31, we fix a subsequence n_k such that $\alpha_J^{n_k}\to\beta\in\Gamma^*$. Comparing characters in (4.58), we have (2.29). We say that $\mathbf{x}_0>\mathbf{a}_0$ is regular for the given β if for a solution of the system (2.29) we have $\mathbf{x}_j\in(\mathbf{a}_j,\mathbf{b}_j)$. If so, we can choose an interval I around $\mathbf{x}_0$ such that all points of this interval are regular ($\mathbf{x}_j\in(\mathbf{a}_j,\mathbf{b}_j)$ depends continuously on $\mathbf{x}_0$). Moreover

$$\inf_{\mathbf{x}_0\in I}\tilde\rho^2(\{\mathbf{x}_j\}_{j=0}^g)>0.$$

Going back to (4.58), for a sufficiently small $\rho^2 : \tilde{\rho}_* < \rho < 0$, we have a unique solution $z_0 = z_0(\mathbf{x}_0, \rho)$ of the equation (2.30),

$$z_0 \simeq \mathbf{x}_0 - i\rho \frac{\sqrt{R_E(\mathbf{x}_0)}}{U'_{\mathfrak{x}}(\mathbf{x}_0)}.$$

To summarize: for an open set of characters $\beta \in \mathcal{V}_1 \subset \Gamma^*$ equations (2.29) and (2.30) set a one-to-one correspondence $z_0 = z_0(\mathbf{x}_0, \rho)$ on an open set $\tilde{\mathcal{V}}_2 = I \times (\tilde{\rho}_*, 0)$. $\mathcal{V}_2 \subset \mathbb{C}_+$ is defined as the image of $\tilde{\mathcal{V}}_2$.

Finally, we note that for an expression of the form

$$g(z) = \frac{z - z_0}{z - \overline{z}_0} f(z), \quad f(\overline{z}_0) = 0,$$

we have

$$|g'(z_0)| = \frac{|f(z_0)|}{2 \operatorname{Im} z_0}.$$

Thus, to get (2.28), we use (4.58) and $\overline{\frac{\sqrt{R_E(z_0)}}{U_{\mathfrak{x}}(z_0)}} = \frac{\sqrt{R_E(\overline{z}_0)}}{U_{\mathfrak{x}}(\overline{z}_0)} = -\frac{i}{\rho}$. □

Let us represent (2.30) in terms of potential theory.

Lemma 4.32. *Let $\omega_{\mathbb{C}_+}(z, F)$ be the harmonic measure of $F \subset \mathbb{R}$ at $z \in \mathbb{C}_+$ in the upper half-plane. Then (2.30) implies*

$$\omega_{\mathbb{C}_+}(z_0, \cup_{j=0}^g [\mathbf{a}_j, \mathbf{x}_j]) = \omega_{\mathbb{C}_+}(z_0, \cup_{j=0}^g [\mathbf{x}_j, \mathbf{b}_j]). \tag{4.60}$$

Moreover, (4.60) means that $\frac{\sqrt{R_E(z_0)}}{U_{\mathfrak{x}}(z_0)}$ assumes a pure imaginary value.

Proof. Due to the integral representation

$$\frac{\sqrt{R_E(z)}}{U_{\mathfrak{x}}(z)} = iCe^{\frac{1}{2}\int_{\cup_{j=0}^g [\mathbf{a}_j, \mathbf{x}_j]} \frac{1+xz}{x-z} \frac{dx}{1+x^2} - \frac{1}{2}\int_{\cup_{j=0}^g [\mathbf{x}_j, \mathbf{b}_j]} \frac{1+xz}{x-z} \frac{dx}{1+x^2}}, \quad C \in \mathbb{R},$$

the required condition $\arg \frac{\sqrt{R_E(z_0)}}{U_{\mathfrak{x}}(z_0)} = \pm\frac{\pi}{2}$ (depending on sign of C) corresponds to

$$\int_{\cup_{j=0}^g [\mathbf{a}_j, \mathbf{x}_j]} \frac{\operatorname{Im} z}{|x - z_0|^2} dx = \int_{\cup_{j=0}^g [\mathbf{x}_j, \mathbf{b}_j]} \frac{\operatorname{Im} z}{|x - z_0|^2} dx$$

that is, to (4.60). □

Theorem 2.20 reduces asymptotics in the complex polynomial Ahlfors problem to the following generalized Abel-Jacobi inversion problem, compare subsection 4.2.3, particularly Proposition 4.23.

Problem 8. *For fixed $\beta \in \Gamma^*$ and $z \in \mathbb{C}_+$ solve the system*

$$\sum_{j=0}^g \omega(\mathbf{x}_j, E_k) = \beta_k, \quad k = 1, \dots, g, \tag{4.61}$$

$$\sum_{j=0}^g \left(\int_{\mathbf{a}_j}^{\mathbf{x}_j} + \int_{\mathbf{b}_j}^{\mathbf{x}_j} \right) \frac{dx}{|x - z|^2} = 0, \tag{4.62}$$

where $\mathbf{x}_j \in [\mathbf{a}_j, \mathbf{b}_j]$.

Remark 4.33. We substitute these values in (2.28) to define, the generally speaking multivalued, function $\Upsilon(\lambda,\beta)=Y(z(\lambda),\beta)\left|\frac{dz}{d\lambda}\right|$ responsible for the required asymptotics.

Proposition 4.34. *Problem 8 is locally solvable.*

Proof. We will check that the Jacobian of the system (4.61)-(4.62) does not vanish. To the polynomial $U_{\mathfrak{x}}(z)$ we associate the polynomial $V_{\mathfrak{x}}(z)$ of the form

$$V_{\mathfrak{x}}(z)=\sum_{j=0}^{g}\frac{\sqrt{R_E(\mathbf{x}_j)}}{U'_{\mathfrak{x}}(\mathbf{x}_j)}\frac{U_{\mathfrak{x}}(z)}{z-\mathbf{x}_j}.$$

By this definition

$$\mathfrak{m}_{\mathfrak{x}}(z):=\det\begin{bmatrix}1 & \dots & 1\\ \vdots & \vdots & \vdots\\ \mathbf{x}_0^{g-1} & \dots & \mathbf{x}_g^{g-1}\\ \frac{\sqrt{R_E(\mathbf{x}_0)}}{z-\mathbf{x}_0} & \dots & \frac{\sqrt{R_E(\mathbf{x}_g)}}{z-x_g}\end{bmatrix}=\prod_{0\le k<j\le g}(\mathbf{x}_k-\mathbf{x}_j)\frac{V_{\mathfrak{x}}(z)}{U_{\mathfrak{x}}(z)}.$$

For this reason the Jacobian

$$\det\begin{bmatrix}1 & \dots & 1\\ \vdots & \vdots & \vdots\\ \mathbf{x}_0^{g-1} & \dots & \mathbf{x}_g^{g-1}\\ \frac{\sqrt{R_E(\mathbf{x}_0)}}{z-\mathbf{x}_0}-\frac{\sqrt{R_E(\mathbf{x}_0)}}{\overline{z}-\mathbf{x}_0} & \dots & \frac{\sqrt{R_E(\mathbf{x}_g)}}{z-\mathbf{x}_g}-\frac{\sqrt{R_E(\mathbf{x}_g)}}{\overline{z}-\mathbf{x}_g}\end{bmatrix}=\mathfrak{m}_{\mathfrak{x}}(z)-\overline{\mathfrak{m}_{\mathfrak{x}}(z)}$$

does not vanish for all $z\in\mathbb{C}_+$ and $\mathbf{x}_j\neq\mathbf{x}_k$ for $j\neq k$. □

Proposition 4.35. *In the elliptic case, $g=1$, Problem 8 is globally solvable, but for some β a solution is not unique.*

Proof. We map Ω to the complement of the system of two arcs E_T such that $z_0\mapsto 0$. Let

$$E_T=\{\zeta=e^{ix}:\ x\in[0,2\pi)\setminus(\mathbf{a}_0,\mathbf{b}_0,)\cup(\mathbf{a}_1,\mathbf{b}_1)\},\quad 0=\mathbf{a}_0<\mathbf{b}_0<\mathbf{a}_1<\mathbf{b}_1<2\pi.$$

Since the harmonic measure $\omega_{\mathbb{C}_+}(z_0,F)$ corresponds to the Lebesgue measure on $\mathbb{T}$, (4.60) corresponds to

$$\mathbf{x}_0+\mathbf{x}_1=\frac{\mathbf{b}_0+\mathbf{a}_1+\mathbf{b}_1}{2}:=c. \tag{4.63}$$

WLOG we assume that $\mathbf{b}_0>\mathbf{b}_1-\mathbf{a}_1$. As soon as $\mathbf{x}_1\in(\mathbf{a}_1,\mathbf{b}_1)$, by (4.63) $\mathbf{x}_0$ runs in the interval (ξ_-,ξ_+), $\xi_+=c-\mathbf{a}_1$, $\xi_-=c-\mathbf{b}_1$, and we have

$$\omega\left(e^{i\xi_+},E_2\right)\le\omega(e^{i\mathbf{x}_0},E_2)+\omega(e^{i\mathbf{x}_1},E_2)\le 1+\omega\left(e^{i\xi_-},E_2\right),\quad E_2=E_T\setminus E_T^1.$$

Since $\omega\left(e^{i\xi_+},E_2\right)<\omega\left(e^{i\xi_-},E_2\right)$, (2.29) is solvable for all $\beta\in\mathbb{R}/\mathbb{Z}$, but in the range $\left(\omega\left(e^{i\xi_+},E_2\right),\omega\left(e^{i\xi_-},E_2\right)\right)$ a solution is not unique. □

Remark 4.36. As it was demonstrated in the previous remark our function $Y(z,\beta)$ describes asymptotics (2.28) for Problems 4, 5 only in a certain vicinity of the real axis. Note, however, that an analytic reproducing kernel is uniquely defined by its diagonal

$$k(\lambda_1,\lambda_2)=\sum\frac{1}{n!m!}(\partial^n\overline{\partial}^m k)(\lambda_0,\lambda_0)(\lambda_1-\lambda_0)^n(\overline{\lambda}_2-\overline{\lambda}_0)^m. \tag{4.64}$$

Therefore, in the case that Conjecture 1 is correct, our Theorem 2.20, in fact, gives a complete information on the asymptotics due to the analytic continuation. Indeed, Theorem 2.19 provides asymptotics on the whole real axis in the z-plane, $\Upsilon(\lambda,\beta)=\frac{1}{2}\mathcal{A}(\lambda,\beta)$, $z(\lambda)\in\mathbb{R}$. To use (4.64) it is enough to have an extension of $\frac{1}{2}\mathcal{A}(\lambda,\beta)$ given by $\Upsilon(\lambda,\beta)$ in an arbitrary small vicinity, $z(\lambda)\in\mathcal{V}_2\subset\mathbb{C}_+$,

$$k^{\alpha(\beta,\lambda_0)}(\lambda_1,\lambda_2)=\sum\frac{1}{n!m!}(\partial^n\overline{\partial}^m\Upsilon)(\lambda_0,\beta)(\lambda_1-\lambda_0)^n(\overline{\lambda}_2-\overline{\lambda}_0)^m,\quad z(\lambda_0)\in\mathbb{R}.$$

Proposition 4.37. *For simply connected domains Problem 8 is uniquely solvable. Moreover,*

$$\Upsilon(\lambda)=Y(z(\lambda))\left|\frac{dz}{d\lambda}\right|=\frac{2\sqrt{\lambda}\overline{\sqrt{\lambda}}}{(\lambda+\overline{\lambda})(\sqrt{\lambda}+\overline{\sqrt{\lambda}})^2}. \tag{4.65}$$

Proof of Proposition 4.37 and Theorem 2.22. By

$$w(z)=\frac{1+z}{1-z}$$

We map $\overline{\mathbb{C}}\setminus[-1,1]$ onto $\mathbb{C}\setminus\mathbb{R}_+$. Hence,

$$\frac{U_{\mathbf{x}_0}(z_0)}{\sqrt{T(z_0)}}=-i\rho$$

is equivalent to

$$\operatorname{Re}\frac{-w_0+\xi_0}{\sqrt{-w_0}}=0,$$

where $w_0=w(z_0)$ and $\xi_0=w(\mathbf{x}_0)$. That is,

$$\operatorname{Re}\sqrt{-w_0}+\xi_0\frac{\operatorname{Re}\sqrt{-w_0}}{|w_0|}=0.$$

Thus, we get a unique $\xi_0=-|w_0|$. By (2.28), we have

$$Y(w_0)=\frac{1}{|w_0-\overline{w}_0|}\left|\frac{\sqrt{-w_0}-\sqrt{|w_0|}}{\sqrt{-w_0}+\sqrt{|w_0|}}\right|.$$

We consider the right half-plane as the universal covering. Let $w=-\lambda^2$. Then

$$Y(w_0)=\left|\frac{1}{(\lambda-\overline{\lambda})(\lambda+\overline{\lambda})}\frac{\lambda-|\lambda|}{\lambda+|\lambda|}\right|=\frac{1}{(\lambda+\overline{\lambda})(\sqrt{\lambda}+\overline{\sqrt{\lambda}})^2}.$$

Since $|w'(\lambda)|=2|\lambda|$, we get (4.65).

Since this solution is global the argument in the proof of Theorem 2.20 are applicable for all $z_0\in\mathbb{C}_+$, which concludes the proof.

□

4.2.6 Adaptations for E_T

We would like to mention some adaptations for an analog result of Theorem 2.20 for vicinities of $\mathbb{T} \setminus E_T$. Let us fix from the very beginning the gap $U_0 = \{\zeta = e^{iz} : z \in (\mathbf{a}_0, \mathbf{b}_0)\}$. Due to symmetry, it is again enough to consider the problem for $|\zeta_0| < 1$ and the Kolmogorov criterion reads

$$\min_{\mathcal{Z}} \operatorname{Re}(\zeta - \zeta_0)^2 Q_{n-2}(\zeta)\overline{P_n(\zeta)} \leq 0, \tag{4.66}$$

where P_n is the extremal polynomials, Q_{n-2} is an arbitrary polynomial of degree $n-2$ and $\mathcal{Z} = \{\zeta \in E_T : \ |P_n(\zeta)| = 1\}$. We define the polynomial

$$S(\zeta) = \frac{1 - \overline{\zeta}_0 \zeta}{\zeta - \zeta_0} P_n(\zeta).$$

Clearly, all zeros of S lie outside of $\mathbb{D}$, for if $|\zeta_1| < 1$ would be another zero of S, then

$$\tilde{P}_n(\zeta) = \frac{1 - \overline{\zeta}_1 \zeta}{\zeta - \zeta_1} P_n(\zeta).$$

would still be in the considered class for Problem 5 but would satisfy $|\tilde{P}_n'(\zeta_0)| > |P_n'(\zeta_0)|$. For arbitrary Q_{n-2} we define

$$F(\zeta) = \zeta^{-\frac{n}{2}} S(\zeta) \quad \text{and} \quad G(\zeta) = \zeta^{-\frac{n}{2}+1} Q_{n-2}(\zeta).$$

Now let

$$F(\zeta) = \Phi(\zeta) + i\Psi(\zeta), \quad G(\zeta) = \mathcal{X}(\zeta) + i\mathcal{Y}(\zeta).$$

the decomposition into real and imaginary part of these functions, which is defined by

$$\Phi(\zeta) = \frac{1}{2}(F(\zeta) + \overline{F(1/\overline{\zeta})}), \quad \Psi(\zeta) = \frac{1}{2i}(F(\zeta) - \overline{F(1/\overline{\zeta})})$$

and correspondingly for G. Taking into account that

$$\frac{(e^{iz} - e^{i\mathbf{u}})(e^{iz} - e^{i\mathbf{v}})}{(e^{iz} - e^{i\mathbf{a}})(e^{iz} - e^{i\mathbf{b}})} = e^{i\frac{\mathbf{u}+\mathbf{v}-(\mathbf{a}+\mathbf{b})}{2}} \frac{\sin(\frac{z-\mathbf{u}}{2})\sin(\frac{z-\mathbf{v}}{2})}{\sin(\frac{z-\mathbf{a}}{2})\sin(\frac{z-\mathbf{b}}{2})},$$

one gets in a complete way a corresponding result to Theorem 2.20 for vicinities of $\mathbb{T} \setminus E_T$.

Example 4.38. Numerical experiments show that already in the elliptic case a certain bifurcation in the shape of an extremal polynomial may occur. In this example we use the notation for elliptic functions from [8]. We fix a set $E = [-1/k, 1/k] \setminus (-k, k)$, for some $k < 1$ and fix $z_0 \in \mathbb{C}_+$. Define $\zeta : \mathbb{C}_+ \to \mathbb{D}$ by

$$\zeta(z) = \frac{1/k - \overline{z_0}}{1/k - z_0} \frac{z - z_0}{z - \overline{z}_0}.$$

Let us assume $\xi_1 \in (-k, k)$ and $\xi_0 \in \mathbb{R} \setminus [-1/k, 1/k]$ form the set $\mathfrak{X}$. By (4.63), $\zeta(\xi_1)$, depending on ξ_0, is given by

$$\zeta_1 = \zeta_1(\xi_0) = \frac{\sqrt{\zeta(-1/k)\zeta(-k)\zeta(k)\zeta(1/k)}}{\zeta(\xi_0)},$$

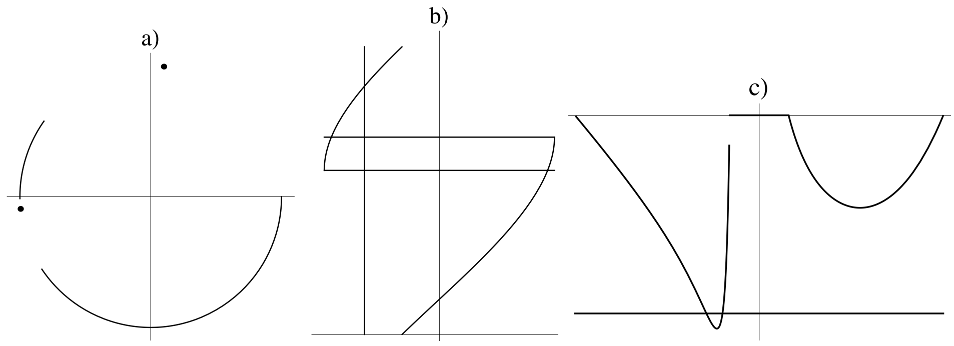


Figure 4.4: A modification of the shape of an extremal polynomial is required

and

$$\xi_1(\xi_0) = \frac{z_0 - \overline{z}_0 \zeta_1}{1 - \zeta_1}.$$

In Figure 4.4 we implemented these calculations for $k = 0.4$ and $z_0 = \frac{2}{\sqrt{2}}(1+i)$, varying ξ_0. In a) one can see $E_T = \zeta(E)$ and the points ζ_0 and ζ_1. The elliptic integral of the first kind

$$\lambda(z) = \int_0^z \frac{dx}{\sqrt{(k^2 - x^2)(1 - k^2 x^2)}},$$

maps $\Omega = \overline{\mathbb{C}} \setminus E$ onto a rectangle

$$\Delta = \{\lambda : \ 0 < \operatorname{Re}\lambda < K, \ -iK' < \operatorname{Im} < iK'\}.$$

That is,

$$z = k \operatorname{sn}(\lambda; k^2).$$

Due to Lemma 1.5, we can compute the harmonic measure of $I_+ = [k, 1/k]$ explicitly. Namely,

$$\omega(z, I_, \Omega) = \operatorname{Re}\left(\frac{\lambda + K}{2K}\right)$$

Therefore, we can compute

$$\beta(\xi) := \omega(\xi, I_+; \Omega) + \omega(\xi_1(\xi), I_+; \Omega) \mod 1$$

explicitly. Plot b) shows $\beta(\xi)$, and the intersection of the vertical line with $\beta(\xi)$ indicates the value $\beta(\xi_0)$ corresponding to Plot a). In particular we see that it is in a range where the solution of (4.61) is unique. We write again

$$E_T = \{\zeta = e^{ix} : \ x \in [0, 2\pi) \setminus (0, \mathbf{b}_0,) \cup (\mathbf{a}_1, \mathbf{b}_1)\}, \quad 0 < \mathbf{b}_0 < \mathbf{a}_1 < \mathbf{b}_1 < 2\pi.$$

Due to (4.63) and (2.30) we see that

$$\frac{1}{\rho^2} = \frac{\mathcal{R}_{E_T}(0)}{U_{\mathfrak{X}}(0)^2} = 1.$$

In c) we plot $-\frac{\mathcal{R}_{E_T}(\zeta)}{U_{\mathfrak{X}}(\zeta)^2}$, $\zeta \in E_T$, for $\mathfrak{X}_T = \{\zeta_0, \zeta_1\}$. We see that

$$-\min_{\zeta \in E_T} \frac{\mathcal{R}_{E_T}(\zeta)}{U_{\mathfrak{X}}(\zeta)^2} < -1.$$

Therefore, the value ρ violates (4.54).

Appendix A

Appendix

A.1 Hyperelliptic Riemann surfaces

This section is mainly based on the very pointed presentation in [12], specified on the setting of hyperelliptic Riemann surfaces. Let $-\infty < \mathbf{b}_0 < \mathbf{a}_1 < \mathbf{b}_1 < \cdots < \mathbf{b}_g < \mathbf{a}_0 < \infty$ and the associated polynomial

$$R_E(z) = \prod_{i=0}^{g} (z - \mathbf{a}_i)(z - \mathbf{b}_i).$$

To this collection we associate the set of points

$$\tilde{\mathcal{R}} = \{p = (z, w) : w^2 = R_E(z)\},$$

endowed with the topology from $\mathbb{C}^2$. It is compactified by adding two points $\infty_\pm$. It is usually visualized by gluing together two copies of $\Omega = \overline{\mathbb{C}} \setminus E$, where $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{i=1}^{g} (\mathbf{a}_i, \mathbf{b}_i)$. We agree to fix the upper sheet, $\mathcal{R}_+$ by the condition that $w > 0$ for $z > \mathbf{a}_0$. The two additional points $\infty_\pm$ correspond to the two infinities of the upper and lower sheet $\mathcal{R}_\pm$. We will denote this set by $\mathcal{R}$. In fact it is a compact Riemann surface. This means, that for each point $p \in \mathcal{R}$, there exists a vicinities $\mathcal{U}$ of p and $\mathcal{V} \subset \mathbb{C}$ of 0 and a homeomorphism

$$\varphi : \mathcal{U} \to \mathcal{V}.$$

We will call the mapping φ *local coordinates* of the point p. Moreover, this local coordinates are analytically compatible, i.e., for any two local coordinates $\varphi, \tilde{\varphi}$, with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, the transition functions

$$\varphi \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(\mathcal{U} \cap \tilde{\mathcal{U}}) \to \varphi(\mathcal{U} \cap \tilde{\mathcal{U}})$$

are analytic. A set $\{\varphi_i\}$ of analytically compatible local coordinates such that the collection of corresponding vicinities $\{\mathcal{U}_i\}$ cover the whole manifold is called an *atlas* of the Riemann surface. Since $\operatorname{grad}(R_E(z) - w^2)$ is not vanishing on $\mathcal{R}$, the implicit function theorem guarantees the existence of an atlas. In fact, the local coordinates can be given explicitly. The set of points $\{\mathbf{a}_i, \mathbf{b}_i\}$ is called branch points. If for $p_0 = (z_0, w_0)$, z is not a branch point and $p_0 \neq \infty_\pm$ the local coordinates are given by

$$(z, w) \mapsto z - z_0,$$

and if p_0 is a branch point they are given by

$$(z, w) \mapsto \sqrt{z - z_0}.$$

The local coordinates of infinity are $(z, w) \mapsto 1/z$. A function $f : \mathcal{R} \to \overline{\mathbb{C}}$ is called meromorphic, if

$$f(z) = f(\varphi^{-1}(z)),$$

is meromorphic in any local coordinates φ on $\varphi(\mathcal{U})$. Any meromorphic function on $\mathcal{R}$ can be written as

$$f(p) = \frac{A(z)w + B(z)}{C(z)},$$

where A, B, C are polynomials. An *Abelian differential* $\mathrm{d}j$ on $\mathcal{R}$ is a meromorphic 1-form on $\mathcal{R}$, i.e., locally it can be written as

$$f(z)\mathrm{d}z$$

for a meromorphic function f. An Abelian differential is of the

(i) *first kind*, if f is analytic,

(ii) *second kind*, if $\mathrm{Res}_{p_0}\,\mathrm{d}j = 0$ for every $p_0 \in \mathcal{R}$,

(iii) *third kind* else.

It is a general result of Riemann that the dimension of the space of Abelian differentials of the first kind coincides with the genus of the surface. For $\mathcal{R}$ one can show that every Abelian differential of the first kind, $\mathrm{d}j_1$, is of the form

$$dj_1(p) = \frac{P(z)}{w}\mathrm{d}z,$$

where $P(z)$ is a polynomial of degree at most $g - 1$. We fix a canonical basis

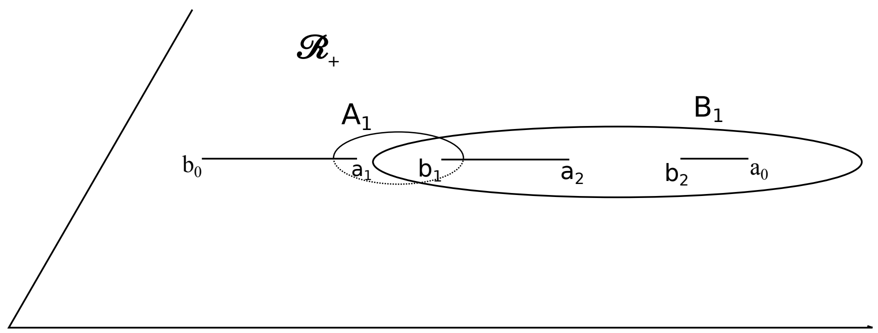


Figure A.1: A and B cuts

of cycles on $\mathcal{R}$. Let $E_k = [\mathbf{b}_i, \mathbf{a}_0] \cap E$, A_i be contours encircling the gap $(\mathbf{a}_i, \mathbf{b}_i)$ clockwise and B_i be a contour on $\mathcal{R}_+$ that starts at ∞_+ goes into the upper half plane passes through the gap $(\mathbf{a}_i, \mathbf{b}_i)$ and returns on the lower half-plane such that E_k is in its interior; cf. Figure A.1. To any Abelian differential, we can associate an Abelian integral, by

$$J(p) = \int_{p_0}^{p} \mathrm{d}j,$$

where p_0 is some fixed base point on $\mathcal{R}$. $J(p)$ is a multi-valued function on $\mathcal{R}$. For a basis of Abelian differential of the first kind,$\{\mathrm{d}j_k^1\}_{k=1}^g$, we define its matrix of A- and B-periods by

$$\Pi_{l,k}^A = \int_{A_k} \mathrm{d}j_1^l, \quad \Pi_{l,k}^B = \int_{B_k} \mathrm{d}j_1^l.$$

By Stoke's theorem, one can show that

$$\mathrm{Im}(\Pi^A)^* \Pi^B > 0,$$

which implies that Π^A is invertible. Hence, we can choose a basis $\{\mathrm{d}j_1^l\}$ such that Π^A is the identity matrix. This basis is of the form

$$\mathrm{d}j_1^l(p) = C_l \frac{\prod_{i\neq l}(z - \mathbf{x}_i)}{w} \mathrm{d}z,$$

where

$$\mathbf{x}_i \in (\mathbf{a}_i, \mathbf{b}_i), \quad C_l^{-1} = \int_{\mathbf{a}_l}^{\mathbf{b}_l} \frac{\prod_{i\neq l}(z - \mathbf{x}_i)}{w} \mathrm{d}z.$$

We define the *Abel map* $\mathcal{A} : \mathcal{R} \to \mathbb{C}^g/L$, where $L = I + \Pi^B$, by

$$\mathcal{A}(p)_l = \int_{p_0}^{p} \mathrm{d}j_1^l.$$

One can show that $\mathcal{A}$ is well defined. A divisor D on a Riemann surface is a formal finite sum

$$D = \sum_i n_i p_i, \quad n_i \in \mathbb{Z}, p_i \in \mathcal{R}.$$

$\mathcal{A}$ is linearly extended to the set of divisors. The set of divisors, $\mathrm{Div}(\mathcal{R})$, forms an Abelian group with respect to the naturally defined operation of addition. The degree of a divisor, $D = \sum_i n_i p_i$, is defined by

$$\deg D = \sum_i n_i.$$

The kernel of deg is called the *Picard group* of the Riemann surface, i.e., $\mathrm{Pic}(\mathcal{R}) = [\deg = 0]$. A divisor of a meromorphic function f is the sum of its zeros and poles, where $n_i > 0$ denotes the multiplicity of a zero and $n_i < 0$ denotes the multiplicity of a pole. It is denoted by (f). A divisor, D, is called principle if there exists a function such that $(f) = D$. Let $\mathrm{Pr}(\mathcal{R})$ be the subgroup of principle divisors. Finally the Jacobian of $\mathcal{R}$ is defined as

$$\mathrm{Jac}(\mathcal{R}) = \mathrm{Pic}(\mathcal{R}) / \mathrm{Pr}(\mathcal{R}).$$

We are aware that it is quite common in the literature that $\mathbb{C}^g/L$ is called the Jacobian variety. We follow the notation of Mumford [45].

Theorem A.1 (Abel)**.** *The Abel map*

$$\mathcal{A} : \mathrm{Jac}(\mathcal{R}) \to \mathbb{C}^g/L$$

is injective.

To show this, one has to construct for every divisor with the property that $\mathcal{A}(D) \in L$ a meromorphic function, f, such that $(f) = D$. The basic idea is the following. Riemann showed that for any finite set of points $\{p_i\}$, there exists an Abelian differential which is analytic on $\mathcal{R} \setminus \{p_i\}$ with arbitrary preassigned principle part. Hence, writing

$$D = \sum_i (p_i - d_i),$$

which is possible since $\deg D = 0$, we can define Abelian differentials of the third kind $\mathrm{d}j_3^i$ such that it is analytic on $\mathcal{R} \setminus \{p_i, d_i\}$ and has

$$\mathrm{Res}_{p_i} \mathrm{d}j_3^i = 1, \quad \mathrm{Res}_{d_i} \mathrm{d}j_3^i = -1.$$

They are normalized by the condition that the A-periods vanish. If we assume that

$$\mathcal{A}(D) = 0,$$

not in the sense of $\mathbb{C}^g/L$, then

$$f = e^{\sum_i \int_{p_0}^{p} \mathrm{d}j_3^i},$$

does the job. Otherwise a certain rescaling is required. In fact, $\mathcal{A}$ is surjective. We say that a divisor, $D = \sum_i n_i p_i$, is positive if $n_i \geq 0$ for all i.

Theorem A.2. *Let $D_{+,g}$ be the set of positive divisors of degree g. Then*

$$\mathcal{A} : D_{+,g} \to \mathbb{C}^g/L$$

is surjective.

Hence $\mathcal{A} : \mathrm{Jac}(\mathcal{R}) \to \mathbb{C}^g/L$ is bijective.

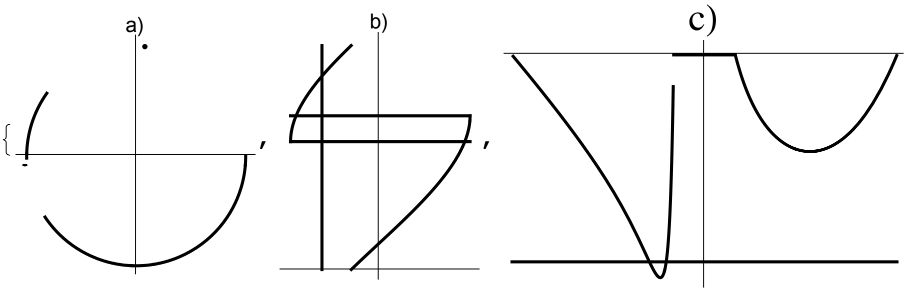
A.2 Addendum to Example 4.38

We have made numerical experiments to example 4.38 using Mathematica.

```
In[168]:= u[z_, z0_, c_] := c * (z - z0) / (z - Conjugate[z0])

In[169]:= v1[z_, z0_, c_, k_] :=
  -Sqrt[u[-1 / k, z0, c] * u[-k, z0, c] * u[k, z0, c] * u[1 / k, z0, c]] / (c * u[z, z0, c])

In[170]:= z1[z_, z0_, c_, k_] := (z0 - Conjugate[z0] * v1[z, z0, c, k]) / (1 - v1[z, z0, c, k])

In[171]:= m1[z_, z0_, c_, k_] := InverseJacobiSN[z1[z, z0, c, k] / k, k^4]

In[172]:= w[z_, k_, K_] := Re[(m[z, k] + K) / (2 K)]

In[173]:= m[z_, k_] := InverseJacobiSN[z / k, k^4]

In[174]:= wS[z_, z0_, c_, k_, K_] := Mod[w[z, k, K] + w[z1[z, z0, c, k], k, K], 1]

In[175]:= wS2[z_, z0_, c_, k_, K_] := w[z, k, K] + w[z1[z, z0, c, k], k, K]

In[176]:= f[z_, z0_, c_, k_, x_] :=
  -((u[z, z0, c] - u[-1 / k, z0, c]) (u[z, z0, c] - u[1 / k, z0, c])
      (u[z, z0, c] - u[-k, z0, c]) (u[z, z0, c] - u[k, z0, c])) /
   ((u[z, z0, c] - u[x, z0, c]) (u[z, z0, c] - c * v1[x, z0, c, k]))^2

In[177]:= g[z_, z0_, c_, k_, x_] :=
  Piecewise[{{f[z, z0, c, k, x], -1 / k ≤ z ≤ -k}, {f[z, z0, c, k, x], k ≤ z ≤ 1 / k}}]

In[178]:= H[kIn_, xIn_, zIn_] :=
  Module[{x = xIn, k = kIn, K, z0 = zIn, c, x2, sol, uR, uI, vR, vI},
   c = (1 / k - Conjugate[z0]) / (1 / k - z0);
   uR = Re[u[x, z0, c]];
   uI = Im[u[x, z0, c]];
   vR = Re[c * v1[x, z0, c, k]];
   vI = Im[c * v1[x, z0, c, k]];
   K = EllipticK[k^4];
   {Show[
     ParametricPlot[
      {{Re[u[(1 - t) (k) + t (1 / k), z0, c]], Im[u[(1 - t) (k) + t (1 / k), z0, c]]},
       {Re[u[(1 - t) (-1 / k) + t (-k), z0, c]], Im[u[(1 - t) (-1 / k) + t (-k), z0, c]]}},
      {t, 0, 1}, PlotStyle → Black, Ticks -> None,
      PlotRange → 1.1, PlotLabel → "a)"],
     Graphics[{PointSize[0.02], Black, Point[{uR, uI}]}],
     Graphics[{PointSize[0.02], Black, Point[{vR, vI}]}]
    ],
    ParametricPlot[{{t, wS[t, z0, c, k, K]}, {t, w[z1[k, z0, c, k], k, K]},
      {t, w[z1[-k, z0, c, k], k, K]}, {x, (k - t) / (2 k)}},
     {t, -k, k}, PlotStyle → Black, Ticks → None, PlotLabel → "b)"]
    , Plot[{g[z, z0, c, k, x], -1}, {z, -1 / k, 1 / k}, PlotStyle → Black, Ticks → None,
     PlotLabel → Text[Style["c)", FontSize → 20]], ImageSize → Medium]
   }
  ]

In[179]:= κ = 0.4; ξ = -0.26; zmax = √2 (1 + ⅈ);
```

In[180]:= H[κ, ξ, z_{max}]

Out[180]= {



}

Bibliography

[1] M. B. Abrahamse, *The Pick interpolation theorem for finitely connected domains*, Michigan Math. J. **26** (1979), no. 2, 195–203.

[2] L. V. Ahlfors, *Bounded analytic functions*, Duke Math. J. **14** (1947), 1–11.

[3] ______, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[4] N. I. Akhiezer, *Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen I.*

[5] ______, *Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen II.*

[6] ______, *Orthogonal polynomials on several intevals*, Dokl. Akad. Nauk SSSR **134** (1960), 9–12 (Russian), translated as Soviet Math. Dokl., **1** (1960), 989–992.

[7] ______, *Orthogonal polynomials on a system of intervals and their continuous analogues*, Proc. Fourth All-Union Math. Congr. (Leningrad, 1961), Vol. 2, "Nauka", Leningrad (1964), 623–628.

[8] ______, *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs, vol. 79, American Mathematical Society, Providence, RI, 1990, Translated from the second Russian edition by H. H. McFaden.

[9] A. I. Aptekarev, *Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda chains*, Math. USSR Sb. **53** (1986), no. 2, 233–260, Russian original in Mat. Sb. (N.S.) **125(167)** (1984), 231–258.

[10] J. A. Ball and V. Vinnikov, *Zero-pole interpolation for meromorphic matrix functions on an algebraic curve and transfer functions of* 2*D systems*, Acta Appl. Math. **45** (1996), no. 3, 239–316.

[11] L. Baratchart, S. Kupin, V. Lunot, and M. Olivi, *Multipoint Schur algorithm and orthogonal rational functions, I: Convergence properties*, J. Anal. Math. **114** (2011), 207–253.

[12] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Verlag, Berlin Heidelberg, 1994.

[13] M. Benedicks, *Positive harmonic functions vanishing on the boundary of certain domains in* $\mathbf{R}^n$, Ark. Mat. **18** (1980), no. 1, 53–72.

[14] S. Bernstein, *Sur la meilleure approximation de* $|x|$ *par des polynomes de degrés donnés*, Acta Math. **37** (1914), no. 1, 1–57.

[15] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56.

[16] J. S. Christiansen, B. Simon, and M. Zinchenko, *Asymptotics of Chebyshev polynomials, I: subsets of* $\mathbb{R}$, Invent. Math. **208** (2017), no. 1, 217–245.

[17] W. Craig, *The trace formula for Schrödinger operators on the line*, Comm. Math. Phys. **126** (1989), no. 2, 379–407.

[18] D. Damanik, R. Killip, and B. Simon, *Perturbations of orthogonal polynomials with periodic recursion coefficients*, Ann. of Math. (2) **171** (2010), no. 3, 1931–2010.

[19] D. Damanik and P. Yuditskii, *Counterexamples to the Kotani-Last conjecture for continuum Schrödinger operators via character-automorphic Hardy spaces*, Adv. Math. **293** (2016), 738–781.

[20] P. L. Duren, *Theory of* H^p *spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.

[21] B. Eichinger, *Periodic GMP matrices*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 066, 19.

[22] ———, *Szegő-Widom asymptotics of Chebyshev polynomials on circular arcs*, J. Approx. Theory **217** (2017), 15–25.

[23] B. Eichinger, F. Puchhammer, and P. Yuditskii, *Jacobi flow on SMP Matrices and Killip–Simon Problem on Two Disjoint Intervals*, Computational Methods and Function Theory (2014), 1–39.

[24] B. Eichinger and P. Yuditskii, *Ahlfors problem for polynomials*, arXiv:1612.02949, 29 pages, special issius dedicated to the 150th anniversary of Mat. Sb., to appear.

[25] A. Eremenko and P. Yuditskii, *Comb functions*, Recent advances in orthogonal polynomials, special functions, and their applications, Contemp. Math., vol. 578, Amer. Math. Soc., Providence, RI, 2012, pp. 99–118.

[26] G. Faber, *Über Tschebyscheffsche Polynome*, J. Reine Angew. Math. **150** (1920), 79–106.

[27] P. R. Garabedian, *Schwarz's lemma and the Szegö kernel function*, Trans. Amer. Math. Soc. **67** (1949), 1–35.

[28] J. B. Garnett, *Bounded analytic functions*, first ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.

[29] G. H. Golub and C. F. Van Loan, *Matrix computations*, fourth ed., Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2013.

[30] M. Hasumi, *Hardy classes on infinitely connected Riemann surfaces*, Lecture Notes in Mathematics, vol. 1027, Springer-Verlag, Berlin, 1983.

[31] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[32] E. Hendriksen and C. Nijhuis, *Laurent-Jacobi matrices and the strong Hamburger moment problem*, Proceedings of the International Conference on Rational Approximation, ICRA99 (Antwerp), vol. 61, 2000, pp. 119–132.

[33] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. **158** (2003), no. 1, 253–321.

[34] A. N. Kolmogorov, *A remark on the polynomials of P. L. Čebyšev deviating the least from a given function*, Uspehi Matem. Nauk (N.S.) **3** (1948), no. 1(23), 216–221.

[35] M. G. Kreĭ n and A. A. Nudelman, *The Markov moment problem and extremal problems*, American Mathematical Society, Providence, R.I., 1977, Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50.

[36] I. M. Kričever, *Algebraic curves and nonlinear difference equations*, Uspekhi Mat. Nauk **33** (1978), no. 4(202), 215–216.

[37] S. Kupin and P. Yuditskii, *Analogs of the Nehari and Sarason theorems for character-automorphic functions and some related questions*, Topics in interpolation theory, Oper. Theory Adv. Appl., vol. 95, Birkhäuser, 1997, pp. 373–390.

[38] B. Ya. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman.

[39] ———, *The connection of a majorant with a conformal mapping. II*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. (1989), no. 52, 3–21.

[40] ———, *Majorants in classes of subharmonic functions*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. (1989), no. 51, 3–17.

[41] ———, *Lectures on entire functions*, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996, In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.

[42] A. L. Lukashov, *Inequalities for the derivatives of rational functions on several intervals*, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 3, 115–138.

[43] V. A. Marchenko and I. V. Ostrovskii, *A characterization of the spectrum of hill's operator*, Mathematics of the USSR-Sbornik **26** (1975), no. 4, 493.

[44] A. A. Markov, *Lectures on functions deviating least from zero.*, (1906), 244–291, (Selected papers, Gostehizdat, Moscow- Leningrad, 1948, in Russian).

[45] D. Mumford, *Tata lectures on theta. I*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007.

[46] ———, *Tata lectures on theta. II*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007, Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura, Reprint of the 1984 original.

[47] R. Nevanlinna, *Eindeutige analytische Funktionen*, J. W. Edwards, Ann Arbor, Michigan, 1944.

[48] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 297, Springer-Verlag, Berlin, 1992.

[49] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Anal. Math. **89** (2003), 113–154.

[50] C. Pommerenke, *Über die analytische Kapazität*, Archiv der Mathematik **11** (1960), no. 1, 270–277 (German).

[51] ———, *On the Green's function of Fuchsian groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **2** (1976), 409–427.

[52] V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Amer. Math. Soc. Transl. (2) **15** (1960), 131–243.

[53] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.

[54] C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, Ann. of Math. **174** (2011), no. 1, 125–171.

[55] W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc. **78** (1955), 46–66.

[56] B. Simon, *Szegő's theorem and its descendants*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011, Spectral theory for L^2 perturbations of orthogonal polynomials.

[57] ______, *Basic complex analysis*, A Comprehensive Course in Analysis, Part 2A, American Mathematical Society, Providence, RI, 2015.

[58] M. Sodin and P. Yuditskii, *Functions that deviate least from zero on closed subsets of the real axis*, Algebra i Analiz **4** (1992), no. 2, 1–61.

[59] ______, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), no. 3, 387–435.

[60] G. Szegő, *Bemerkungen zu einer Arbeit von Herrn M. Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **21** (1924), no. 1, 203–208.

[61] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, vol. 72, American Mathematical Society, Providence, RI, 2000.

[62] J.-P. Thiran and C. Detaille, *Chebyshev polynomials on circular arcs in the complex plane*, Progress in approximation theory, Academic Press, Boston, MA, 1991, pp. 771–786.

[63] V. Totik, *Chebyshev polynomials on compact sets*, Potential Anal. **40** (2014), no. 4, 511–524.

[64] V. Totik and P. Yuditskii, *On a conjecture of Widom*, J. Approx. Theory **190** (2015), 50–61.

[65] S. Verblunsky, *On Positive Harmonic Functions: A Contribution to the Algebra of Fourier Series*, Proc. London Math. Soc. **S2-38** (1935), no. 1, 125.

[66] A. Volberg and P. Yuditskii, *Kotani-Last problem and Hardy spaces on surfaces of Widom type*, Invent. Math. **197** (2014), no. 3, 683–740.

[67] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Advances in Math. **3** (1969), 127–232.

[68] ______, *$\mathcal{H}_p$ sections of vector bundles over Riemann surfaces*, Ann. of Math. **94** (1971), 304–324.

[69] E. P. Wigner, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. (2) **67** (1958), 325–327.

[70] P. Yuditskii, *A complex extremal problem of Chebyshev type*, J. Anal. Math. **77** (1999), 207–235.

[71] ______, *A special case of de Branges' theorem on the inverse monodromy problem*, Integral Equations Operator Theory **39** (2001), no. 2, 229–252.

[72] ______, *Killip-Simon problem and Jacobi flow on GMP matrices*, arXiv:1505.00972 (2015).

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch.

Linz, Juli 2017

Benjamin Eichinger

Curriculum Vitae

Benjamin Eichinger

July 2017

Address

Institute for Analysis, Johannes Kepler University Linz, Altenbergerstraße 69, 4040 Linz, Austria

E-mail: `benjamin.eichinger@jku.at`

homepage: `http://www.dynamics-approx.jku.at/eichinger/`

Personal Data

Date of birth: March 2, 1990

Place of birth: Linz, Austria

Citizenship: Austria

Social Service: 2009

Languages: German, English

Research Interests

Spectral Theory of Jacobi Matrices

Functional Models of Operators

Orthogonal Polynomials

Extremal Problems of Chebyshev type.

Education

2014: Diploma in Mathematics, Johannes Kepler University Linz

Diploma Thesis

"Spectral Theory of Jacobi matrices and the Jacobi flow on SMP matrices", May 2014, Johannes Kepler University Linz
Thesis advisor: Univ.Doz. Dr. Peter Yuditskii

Workshops

2014: Masterclass by Barry Simon, Spectral Theory of Orthogonal Polynomials, Aarhus.
2015: Winter school on Canonical System (direct spectral theory) by Michael Kaltenbäck and Harald Woracek, Austria.

Talks

11. *Chebyshev problems on a Circular Arc*, 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications, Kent,Great Britain, July 2017.
10. *Chebyshev problems on a Circular Arc*, Seminar Talk, Rice University, Houston, Texas, February 2017.
9. *Chebyshev problems on a Circular Arc*, 35th Annual Western States Mathematical Physics Meeting, Caltech, Pasadena, California, February 2017.
8. *Elements of J-theory in the spectral theory of GMP matrices*, Workshop on Operator Theory and Indefinite Inner Product Spaces, Vienna, Austria, December 2016.

7. *Szegő-Widom asymptotics of Chebyshev Polynomials on Circular Arcs*, Workshop on Function Approximation, Linz, Austria, December 2016.

6. *Szegő-Widom asymptotics of Chebyshev Polynomials on Circular Arcs*, Seminar Talk, Lund, Sweden, September 2016.

5. *Szegő-Widom asymptotics of Chebyshev Polynomials on Circular Arcs*, Harmonic Analysis, Complex Analysis, Spectral Theory and all that, Bedlewo, Poland, August 2016.

4. *Szegő-Widom asymptotics of Chebyshev Polynomials on Circular Arcs*, Jaen Conference on Approximation Theory, Ubeda, Spain, July 2016.

3. *Periodic GMP matrices*, EquaDiff 2015, Lyon, France, July 2015.

2. *Killip-Simon Problem and Jacobi Flow on GMP matrices*, Seminar Talk, TU Vienna, Austria, June 2015.

1. *Killip-Simon Problem and Jacobi Flow on GMP matrices*, 13th International Symposium on Orthogonal Polynomials, Special Functions and Applications, Gaithersburg, USA, June 2015.

Teaching Experience

Tutorials in Probability Theory and Statistics, Analysis, Dynamical Systems and Chaos and Functional Analysis in the years 2013-2017 at Johannes Kepler University, Linz.

Publications

4. *Ahlfors problem for polynomials*, with P. Yuditskii, special issues dedicated to the 150th anniversary of Mat. Sb., to appear, arXiv:1612.02949, 30 pages.

3. *Szegő-Widom asymptotics of Chebyshev polynomials on Circular Arcs*, J. Approx. Theory, to appear.

2. *Periodic GMP matrices*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), 066, 19 pages.

1. *Jacobi Flow on SMP Matrices and Killip-Simon Problem on Two Disjoint Intervals*, with F. Puchhammer and P. Yuditskii, Compt. Methods Funct. Theory **16** (2016), no.1, 3-41.