LECTURE NOTES

Introduction to Hardy Spaces

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CHAPTER 1

Introduction

We will introduce the Hardy space $H^2(\mathbb{D}) = H^2$ via an unitary mapping from $\ell^2(\mathbb{N}_0)$. We will justify later, that this definition coincides with the standard definition of Hardy spaces.

Let

$$\ell^2(\mathbb{N}_0) = \left\{ (a_n)_{n=0}^{\infty} \mid a_n \in \mathbb{C}, \sum |a_n|^2 < \infty \right\}.$$

Note that for $(a_n)_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ and $z \in \mathbb{D} = B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$ we have the following estimate

$$\sum_{n=0}^{\infty} |a_n z^n| \leq \|(a_n)\|_{\ell^2} \frac{1}{\sqrt{1 - |z|^2}}.$$

Let $A(\mathbb{D})$ denote the set of analytic functions on $\mathbb{D}$ and define

$$\Phi := \begin{cases} \ell^2(\mathbb{N}_0) & \to \quad A(\mathbb{D}) \\ (a_n) & \mapsto \quad \sum_{n=0}^{\infty} a_n z^n. \end{cases}$$

Then we define

$$H^2(\mathbb{D}) := \Phi(\ell^2(\mathbb{N}_0)).$$

We equip $H^2(\mathbb{D})$ with a Hilbert space structure in the following way. Let $(a_n), (b_n) \in \ell^2$ and define

$$\langle \Phi((a_n)), \Phi((b_n)) \rangle_{H_2} := \langle (a_n), (b_n) \rangle_{\ell_2} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Thus, Φ is per definitionem a unitary map.

Lemma 1.1. *$H^2(\mathbb{D})$ is a reproducing kernel Hilbert space, with reproducing kernel*

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}, \quad \lambda \in \mathbb{D}.$$

Proof. That $H^2(\mathbb{D})$ is a Hilbert space follows from the defintion. It remains to show that $k_\lambda(z)$ is a reproducing kernel. First, let us check that $k_\lambda \in H^2$. We have

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n.$$

That is, $k_\lambda = \Phi((\overline{\lambda}^n)_{n=0}^{\infty})$, and

$$\|(\overline{\lambda}^n)_{n=0}^{\infty}\|_{\ell^2(\mathbb{N}_0)}^2 = \frac{1}{1 - |\lambda|^2}.$$

Let $f = \Phi((a_n)_{n=0}^{\infty})$. Then

$$f(\lambda) = \Phi((a_n)_{n=0}^{\infty})(\lambda)$$

$$= \sum_{n=0}^{\infty} a_n \lambda^n$$
$$= \langle (a_n), (\overline{\lambda}^n) \rangle_{\ell^2}$$
$$= \langle \Phi((a_n)), \Phi((\overline{\lambda}^n)) \rangle_{H^2} = \langle \Phi((a_n)), k_\lambda \rangle_{H^2}$$

This finishes the proof. □

Recall that the right shift S acts on sequences by

$$S := \begin{cases} \ell^2(\mathbb{N}_0) & \to \quad \ell^2(\mathbb{N}_0) \\ (a_0, a_1, \cdots) & \mapsto \quad (0, a_0, a_1, \cdots) \end{cases}$$

Note that $\Phi : \ell^2 \to H^2$ is by construction unitary.

LEMMA 1.2. *Let S denote the right shift on $\ell^2(\mathbb{N}_0)$ and S^* its adjoint. Then we have for every $f \in H^2(\mathbb{D})$*

$$\Phi S \Phi^* f = z f(z)$$

and

$$\Phi S^* \Phi^* f = \frac{f - f(0)}{z} \tag{1.1}$$

PROOF. Exercise □

Because of this relation it is also common to denote the mulitplication operator M_z in $H^2(\mathbb{D})$ by S and the operator in (1.1) by S^* and we will follow this convention in the following.

PROPOSITION 1.3. *Let S denote the unilateral shift on H^2. Then it holds that*

(i) S is an isometry;
(ii) The spectrum of S is $\overline{\mathbb{D}}$;
(iii) S has no eigenvalues;

PROOF. (i): is clear.
(ii): Since $\|S\| = 1$, it follows that $\sigma(S) \subset \overline{\mathbb{D}}$. To prove the other inclusion, note that for all $\lambda \in \mathbb{D}$

$$(S - \lambda)H^2 \subset \{f \in H^2 \mid f(\lambda) = 0\} \neq H^2.$$

and thus, $(S - \lambda)$ cannot be surjective. In fact, $\overline{(S - \lambda)H^2} \subset \{f \in H^2 \mid f(\lambda) = 0\}$ showing, that all $\lambda \in \mathbb{D}$ lie in the residual spectrum of S. Assume that $f_n \in (S - \lambda)H^2$ and $f_n \to f \in H^2$. Thus, $f_n(\lambda) = 0$. On the other hand

$$|f_n(\lambda) - f(\lambda)| = |\langle f_n - f, k_\lambda \rangle \leq \|f_n - f\| \|k_\lambda\| \to 0,$$

showing that $f \in \{f \in H^2 \mid f(\lambda) = 0\}$.
(iii): Assume that for $\lambda \in \mathbb{C}$, there exists $f \in H^2$ so that

$$(S - \lambda)f = 0.$$

That is, $(z - \lambda)f(z) = 0$ and thus $f = 0$ on $\mathbb{D} \setminus \{\lambda\}$ and thus by continuity $f \equiv 0$. This is a contradiction. □

Note that in the proof we have shown the following lemma:

Lemma 1.4. *Let Ω be a set and H be a reproducing kernel Hilbert space over Ω. Then $f_n \to f$ in H implies that $f_n \to f$ pointwise.*

Proposition 1.5. *Let S^* denote the backward shift on H^2. Then every $\lambda \in \mathbb{D}$ is an eigenvalue and it holds that*

$$\overline{\lambda} k_\lambda = S^* k_\lambda$$

Proof. It can be check directly. Alternatively, let $f \in H^2$ and consider

$$\langle f, \overline{\lambda} k_\lambda \rangle = \lambda \langle f, k_\lambda \rangle = \lambda f(\lambda).$$

On the other hand,

$$\langle f, S^* k_\lambda \rangle = \langle Sf, k_\lambda \rangle = \langle z f(z), k_\lambda \rangle = \lambda f(\lambda).$$

Since f was arbitrary, the conclusion follows. □

Thus, we see that studying spectral properties of certain Hilbert space contractions, can be done by studying certain operators on H^2. This is one of the reasons, why Hardy spaces became widely used in spectral theory.

CHAPTER 2

Preliminaries

2.1. Some highlights in Complex Analysis

We start with recalling and providing some new theorems in complex analysis.

DEFINITION 2.1. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. f is called differentiable in $z_0 \in \Omega$, if*

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If f is differentiable for any $z_0 \in \Omega$, then we say that f is holomorphic in Ω. We denote by $\mathrm{Hol}(\Omega)$ *the set of holomorphic functions in Ω, i.e.,*

$$\mathrm{Hol}(\Omega) := \{f \mid f \text{ is holomorphic in } \Omega\}.$$

Note that we do not assume that f' is continuous. This assumption can be removed by Goursat's lemma, which is given in the appendix A.1.

If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma : [0,1] \to \Omega$ a rectifiable curve, then we denote by

$$\int_\gamma f(z) dz$$

the contour integral of f along γ. If $\gamma \in C^1([0,1])$ (or merely piecewise continuously differentiable), it holds that

$$\int_\gamma f(z) dz := \int_0^1 f(\gamma(t)) d\gamma(t) = \int_0^1 f(\gamma(t))\gamma'(t) dt.$$

We now formulate Cauchy's theorem for convex sets.

THEOREM 2.2. *Suppose Ω is a convex open set, $z_* \in \Omega$, f continuous in Ω, and $f \in \mathrm{Hol}(\Omega \setminus \{z_*\})$. Then $f = F'$ for some $F \in \mathrm{Hol}(\Omega)$. Hence, for any closed, rectifiable curve γ*

$$\int_\gamma f dz = 0.$$

PROOF. We sketch the proof. For $a, b \in \Omega$, let $[a,b] = \{at + (1-t)b \mid t \in [0,1]\}$. Fix $z_* \in \Omega$. Then we define

$$F(z) = \int_{[z_*, z]} f(\zeta) d\zeta.$$

By Goursat's Lemma,

$$F(z) - F(z_0) = \int_{[z_0, z]} f(\zeta) d\zeta.$$

Now, standard estimates show that

$$\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$$

and thus, $F' = f$. □

As a consequence, we obtain Cauchy's integral formula:

THEOREM 2.3. *Let Ω and r, z_0 such that $\overline{B_r(z_0)} \subset \Omega$. For every $w \in B_r(z_0)$ and every holomorphic f we have*

$$\int_{|z-z_0|=r} \frac{f(z)}{z-w} dz = 2\pi i f(w).$$

From Theorem 2.3, we obtain Cauchy's integral formula and the fact that holomorphic functions are analytic.

THEOREM 2.4. *If f is holomorphic in $B_R(z_0)$ for some $R > 0$, then f is analytic in z_0 with radius of convergence at least R. That is, f can be represented as a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall z \in B_R(z_0).$$

Moreover, the coefficients are given explicitly by

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

In particular f is n-times differentiable for any $n \in \mathbb{N}$ and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Thus, from now on we will use analytic and holomorphic interchangeably.

The following Cauchy estimate follows:

THEOREM 2.5. *If f is holomorphic in a domain containing $\overline{B_r(z_0)}$ and $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ its power series representation at z_0, then*

$$|a_n| \le \frac{1}{r^n} \int_0^{2\pi} |f(z_0 + re^{it})| \frac{dt}{2\pi}.$$

In particular, if $|f(z)| \le M$ for all $z \in B_r(z_0)$, then

$$|f^{(n)}(z_0)| \le \frac{n!}{r^n} M. \tag{2.1}$$

PROOF. We have

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Using $\gamma(t) = z_0 + re^{it}$, yields

$$|a_n| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0+re^{it})|}{r^{n+1}} r dt.$$

□

Since zero sets of analytic functions are discrete, we obtain the following identity theorem. Recall that $\Omega \subset \mathbb{C}$ is called a region, if Ω is connected and open.

THEOREM 2.6. *Let Ω be a region and $f : \Omega \to \mathbb{C}$ be holomorphic. Then, the following are equivalent:*

(i) $f \equiv 0$;
(ii) there exists $z_0 \in \Omega$ *such that for all* $n \in \mathbb{N}$ $f^{(n)}(z_0) = 0$;
(iii) the set $\{z \in \Omega \mid f(z) = 0\}$ *has an accumulation point in* Ω.

We also recall Morera's theorem, which is a converse to Cauchy's theorem:

THEOREM 2.7 (Morera's theorem). *Let Ω be open and $f : \Omega \to \mathbb{C}$ be continuous. If for any triangle γ with $\operatorname{ins}\gamma \subset \Omega$ it holds that*

$$\int_\gamma f(z)dz = 0, \tag{2.2}$$

then f is holomorphic.

PROOF. Let $z_0 \in \Omega$ and $r > 0$ such that

$$\overline{B_r(z_0)} \subset \Omega. \tag{2.3}$$

Then, as in the proof of Theorem 2.2, we see that f has a local primative $F \in \operatorname{Hol}(B_r(z_0))$ with $F' = f$ there. By Theorem 2.4 this implies that $f \in \operatorname{Hol}(B_r(z_0))$ is holomorphic. Since this holds for any $B_r(z_0)$ satisfying (2.3), we see that $f \in \operatorname{Hol}(\Omega)$. □

REMARK 2.8. *Note that the proof does not yield a global primative, which in general does not exists. Take for instance $f(z) = 1/z$ on $\Omega = B_1(0) \setminus \{0\}$. Domains with the property that every holomorphic function has a primative are sometimes called elementary. It is a crucial theorem in complex analysis, that these are exactly the simply connected domains.*

However, the Cauchy Integral formula yields a global inverse for convex sets. This will be used to show that non vanishing holomorphic functions have "complex logarithm".

THEOREM 2.9. *Let Ω be open and convex, and $f \in \operatorname{Hol}(\Omega)$ such that $f(z) \neq 0$ for all $z \in \Omega$. Then there exists $H \in \operatorname{Hol}(\Omega)$ so that*

$$f = e^H.$$

PROOF. Fix $z_0 \in \Omega$. Since $f(z_0) \neq 0$, there exists $H_0 \in \mathbb{C}$ so that $e^{H_0} = f(z_0)$. Define $h = \frac{f'}{f} \in \operatorname{Hol}(\Omega)$ by assumption. Then, there exists $H \in \operatorname{Hol}(\Omega)$ so that $H' = h$ and $\operatorname{Hol}(z_0) = H_0$. Now consider $g = e^{-H} f$. We have

$$g' = -H' e^{-H} f + e^{-H} f' = e^{-H}(-f' + f') = 0.$$

Thus, g is constant. Since by defintion of H_0, $g(z_0) = 1$, we get $e^H = f$. □

As a consequence of analyticity, we obtain that holomorphic functions satisfy a maximum principle. We will need the following lemma.

LEMMA 2.10. *Let $R > 0, z_0 \in \mathbb{C}$, f be analytic in $B_R(z_0)$ and write*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then, for $0 < r < R$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \int_{-\pi}^{\pi} |f(z_0 + re^{it})|^2 \frac{dt}{2\pi}.$$

PROOF. Fix $0 < r < R$. Then we have

$$f(z_0 + re^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{nit}.$$

Define $g(t) := f(z_0 + re^{it})$ and consider it as a function in $L^2([-\pi, \pi), \frac{d\lambda}{2\pi})$. We will compute its Fourier coefficients. Since the series converges uniformly on $[-\pi, \pi]$, we obtain for $k \in \mathbb{Z}$ that

$$\int_{-\pi}^{\pi} f(z_0 + re^{it}) e^{-ikt} \frac{dt}{2\pi} = \sum_{n=0}^{\infty} a_n r^n \int_{-\pi}^{\pi} e^{nit} e^{-ikt} = a_k r^k.$$

It follows by Parseval's identity that

$$\int_{-\pi}^{\pi} |f(z_0 + re^{it})|^2 \frac{dt}{2\pi} = \int_{-\pi}^{\pi} |g(t)|^2 \frac{dt}{2\pi} = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

□

As a direct consequence, we obtain Liouville's theorem and the maximum principle for holomorphic functions. Recall the following definition:

DEFINITION 2.11. *A function $f \in \mathrm{Hol}(\mathbb{C})$ is called entire.*

THEOREM 2.12 (Liouville's theorem). *Every bounded entire function must be constant.*

PROOF. Suppose f is entire and assume that $f < M$. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$. It follows by Lemma 2.10 that for all $r > 0$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \int_{-\pi}^{\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < M^2$$

It follows that for $a_n = 0$ for $n > 0$ and hence f must be constant. □

As a neat application we can prove the fundamental theorem of Algebra:

THEOREM 2.13 (Fundamental theorem of Algebra). *Every nonconstant polynomial has a zero.*

PROOF. Since P is nonconstant we can write for $N \geq 1$

$$P(z) = \sum_{n=0}^{N} a_n z^n, \quad a_N \neq 0.$$

If P is nonvanishing in all of $\mathbb{C}$, then $f(z) = \frac{1}{P(z)}$ defines a nonvanishing entire function. Since

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|z|^N} = a_N$$

and $N \geq 1$, we get that $\lim_{|z|\to\infty} f(z) = 0$. Thus, by Liouville's theorem $f \equiv 0$. A contradiction. □

THEOREM 2.14 (Maximum modulus principle). *Let Ω be a region and $f \in \mathrm{Hol}(\Omega)$. Then:*

(i) if there exists R, z_0 such that for any $z \in B_R(z_0)$ $|f(z)| \leq |f(z_0)|$, then f is constant.

(ii) if Ω is bounded and f has a continuous extension to $\overline{\Omega}$, then

$$\sup_{z\in\Omega} |f(z)| = \max_{\zeta\in\partial\Omega} |f(\zeta)|.$$

If $z_0 \in \Omega$ with

$$|f(z_0)| = \max_{\zeta\in\partial\Omega} |f(\zeta)|,$$

then f is constant.

PROOF. (i): Assume that for $0 < r < R$ and $t \in [-\pi, \pi]$ it holds that $|f(z_0 + re^{it})| \leq |f(z_0)|$. It follows by Lemma 2.10

$$\int_{-\pi}^{\pi} |f(z_0 + re^{it})|^2 \frac{dt}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq |f(z_0)|^2 = a_0^2.$$

Hence, we conclude that $a_n = 0$ for $n \geq 1$ and thus, f is constant first on $B_R(z_0)$ and then by the identity principle on Ω.

(ii): Let $M = \max_{z\in\overline{\Omega}} |f(\zeta)|$ which exists since $\overline{\Omega}$ is compact. If there exists $z_0 \in \Omega$ such that $|f(z_0)| = M$, then (i) shows that f is constant. Thus, the maximum must be attained at $\partial\Omega$. □

As an application we determine the automorphism group of the unit disc $\mathbb{D} = \{z : \; |z| < 1\}$.

DEFINITION 2.15. *An analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ is called a Schur function.*

LEMMA 2.16. *Let f be a Schur function such that $f(0) = 0$. Then*

$$|f(z)| \leq |z|$$

and $|f'(0)| \leq 1$. If for some $z \in \mathbb{D} \setminus \{0\}$, $|f(z)| = |z|$ or $|f'(0)| = 1$, then $f(z) = cz$ for some unidmodular constant c.

PROOF. Since $f(0) = 0$, the function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{2.4}$$

is analytic in $\mathbb{D}$. For any $r < 1$ we have by the maximum principle that

$$\max_{z:\,|z|\leq r} |g(z)| = \max_{z:\,|z|=r} |g(z)| \leq \frac{1}{r}.$$

Since, this holds for arbitrary $r < 1$, we get $|g(z)| \leq 1$ on $\mathbb{D}$. Since

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z},$$

we conclude that $|f'(0)| \leq 1$. If $g(z)$ attains a maximum in $\mathbb{D}$, we conclude that g is constant. This finishes the proof. $\square$

As an application we are able to characterize the automorphism group of $\mathbb{D}$. Let Ω be a region. An analytic function f on Ω is called an automorphism, if it maps Ω bijectively onto Ω. Since the inverse of an analytic bijection is also analytic (not proved!), the set of automorphism together with composition forms a group. The so-called automorphism group of Ω, $\mathrm{Aut}(\Omega)$.

THEOREM 2.17. *For $z_0 \in \mathbb{D}$ define*

$$\tilde{b}_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Then

$$\mathrm{Aut}(\mathbb{D}) = \{c b_{z_0} : \, z_0 \in \mathbb{D}, |c| = 1\}.$$

PROOF. Note that $\tilde{b}_{z_0}(z)$ has a continuous extension to $\overline{\mathbb{D}}$. For $|z| = 1$ we have $\overline{z} = z^{-1}$ and thus

$$|\tilde{b}_{z_0}(z)| = \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| = \left| \frac{z - z_0}{\overline{z} - \overline{z_0}} \right| = 1$$

Thus, by the maximum principle for $z \in \mathbb{D}$ $|\tilde{b}_{z_0}(z)| < 1$. Since

$$\tilde{b}_{z_0} \circ \tilde{b}_{-z_0}(z) = \frac{\frac{z+z_0}{1+\overline{z_0} z} - z_0}{1 - \frac{z+z_0}{1+\overline{z_0} z} \overline{z_0}} = \frac{(1 - |z_0|^2) z}{1 - |z_0|^2} = z,$$

$\tilde{b}_{z_0} \in \mathrm{Aut}(\mathbb{D})$. Now let $f \in \mathrm{Aut}(\mathbb{D})$. Then there exists $z_0 \in \mathbb{D}$ with $f(z_0) = 0$. Define $g(z) = f \circ b_{-z_0}$. Since $\tilde{b}_{z_0}(z_0) = 0$ and $\tilde{b}_{-z_0} = \tilde{b}_{z_0}^{-1}$, we see that $g(0) = 0$. Moreover, $g \in \mathrm{Aut}(\mathbb{D})$. Define further $h = g^{-1}$. Then also $h(0) = 0$. The Schwarz Lemma implies $|g'(0)|, |h'(0)| \leq 1$. Since $g(h(z)) = z$ we conclude that for $z \in \mathbb{D}$

$$g'(h(z)) h'(z) = 1.$$

Thus in particular $g'(0) h'(0) = 1$ and thus $|g'(0)| = |h'(0)| = 1$. Again, by the Schwarz Lemma we conclude that $g(z) = cz$ and thus $f = c b_{z_0}$. $\square$

2.2. Spaces of analytic functions

In this section assume that Ω is a region. Our goal is to equip the space of analytic functions with a metric such that the $(\mathrm{Hol}(\Omega), d)$ is complete. We see that pointwise convergence is too weak and uniform convergence is too strong. For, one can construct a sequence of analytic functions in $\mathbb{C}$ which converges pointwise to the indicator function of 0, $\chi_{\{0\}}$, which isn't even continuous. On the other hand, $f_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ does not converge uniformly on $\mathbb{C}$ to $\exp(z)$. For $K \subset \mathbb{C}$ compact and f continuous, denote

$$\|f\|_K = \sup_{z \in K} |f(z)|.$$

Definition 2.18. *We denote the set of continuous functions $f : \Omega \to \mathbb{C}$ by $C(\Omega, \mathbb{C})$. We say that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacts, if there exists $f \in C(\Omega, \mathbb{C})$, such that for any $K \subset \Omega$, $\|f_n - f\|_K \to 0$.*

We need to technical lemma in order to define a metric on $C(\Omega, \mathbb{C})$.

Lemma 2.19. *Let Ω be a region. Then there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of Ω such that*

(i) $\cup_{n\in\mathbb{N}} K_n = \Omega$;

(ii) For all $n \geq 1$, $K_n \subset K_{n+1}^\circ$;

Proof. For the case $\Omega = \mathbb{C}$ take $\overline{B_n(0)}$. Otherwise, define Define

$$K_n = \left\{ z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n} \right\} \cap \overline{B_n(0)}.$$

Then K_n is closed as an intersection of two closed sets and bounded and hence compact. Since for every $z \in \Omega$ we have $\operatorname{dist}(z, \mathbb{C} \setminus \Omega) > 0$ and $|z| < \infty$ we have (i). To show (ii) we note that

$$K_n \subset \left\{ z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) > \frac{1}{n+1} \right\} \cap B_{n+1}(0) \subset K_{n+1}$$

and the set in the middle is open. This finishes the proof. □

We call $(K_n)_{n\in\mathbb{N}}$ a compact exhaustion of Ω. For a given compact exhaustion of Ω and $f, g \in C(\Omega, \mathbb{C})$ we define

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}.$$

We have the following Lemma

Lemma 2.20. *We have*

(i) For every $\epsilon > 0$ there $\delta > 0$ and K compact such that for all $f, g \in C(\Omega, \mathbb{C})$ we have

$$\|f - g\|_K < \delta \implies d(f, g) < \epsilon;$$

(ii) For every $\delta > 0$ and K compact, there exists $\epsilon > 0$ such that for all $f, g \in C(\Omega, \mathbb{C})$ we have

$$d(f, g) < \epsilon \implies \|f - g\|_K < \delta;$$

Proof. We start with (i). Choose N so that $\sum_{n=N+1}^{\infty} 2^{-n} \leq \frac{\epsilon}{2}$ and set $K = K_N$ and choose $\delta > 0$ so that $\frac{x}{1+x} < \frac{\epsilon}{2}$ for every $x \in [0, \delta)$. If $\|f - g\|_K < \delta$ we see that

$$\sum_{n=1}^{N} 2^{-n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} < \frac{\epsilon}{2}$$

and the claim follows.

Conversely, let δ and K are given. Since $K \subset \Omega = \cup_{n\in\mathbb{N}} K_n = \cup_{n\in\mathbb{N}} K_n^\circ$, (by compactness) there exists N such that $K \subset K_N$. Choose ϵ such that $\frac{s}{1-s} < \delta$ for $s \in [0, 2^N \epsilon)$. Then

$$\frac{t}{1+t} < 2^N \epsilon \implies t < \delta.$$

If $d(f,g) < \epsilon$ we have

$$\frac{\|f-g\|_{K_N}}{1+\|f-g\|_{K_N}} < 2^N \epsilon$$

and thus $\|f-g\|_K \leq \|f-g\|_{K_N} < \delta$. □

THEOREM 2.21. *Let Ω be a region, $(K_n)_{n\in\mathbb{N}}$ a compact exhaustion and $f_n, f \in C(\Omega,\mathbb{C})$. Then*

(i) $d(f_n,f)\to 0 \iff f_n \to f$ locally on compacts;
(ii) f_n is a Cauchy sequence with respect to d if and only if for every compact set $K\subset\Omega$ f_n is a Cauchy sequence with respect to $\|\cdot\|_K$;
(iii) For every compact set $K\subset\Omega$ $\mathrm{id}: C(\Omega,d)\to C(\Omega,\|\cdot\|_K)$ is continuous.
(iv) The metric space $(C(\Omega,\mathbb{C}),d)$ is complete;

PROOF. (i), (ii) follow directly from Lemma 2.20. (iii) then follows from (i). (iv) follows from the fact, that for any K_n, the space $(C(K_n,\mathbb{C}),\|\cdot\|_{K_n})$ is complete and that the K_n are increasing. □

We have $\mathrm{Hol}(\Omega)\subset C(\Omega,\mathbb{C})$ and thus d also induces a metric on $\mathrm{Hol}(\Omega)$. In this section we consider the metric space $(\mathrm{Hol}(\Omega),d)$.

THEOREM 2.22. *Let Ω be a region. Then*

(i) The space $\mathrm{Hol}(\Omega)$ is a closed subspace of $C(\Omega,\mathbb{C})$. In particular, $\mathrm{Hol}(\Omega)$ is complete;
(ii) If $f_n\to f$ in $\mathrm{Hol}(\Omega)$, then $f_n^{(k)}\to f^{(k)}$ for every $k\in\mathbb{N}$.

PROOF. If $f_n\to f$ in $C(\Omega,\mathbb{C})$, then for every rectifiable γ, $\|f_n-f\|_{\mathrm{ran}\,\gamma}\to 0$. Thus,

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| \leq \ell(\gamma)\|f_n-f\|_{\mathrm{ran}\,\gamma}\to 0.$$

If f_n is a sequence of analytic functions and $f_n\to f$, then for every triangle γ with $\mathrm{ins}\,\gamma\in\Omega$ we have

$$0=\lim_{n\to\infty}\int_\gamma f_n(z)dz=\int_\gamma f(z)dz.$$

Thus, it follows from Morera's theorem that f is analytic and $\mathrm{Hol}(\Omega)$ is closed.

(ii): Let $K\subset\Omega$ be compact and $r>0$ such that

$$E:=\bigcup_{z\in K}\overline{B_r(z)}=\{z\in\mathbb{C}\mid \mathrm{dist}(z,K)\leq r\}$$

is a compact subset of Ω. Hence, for every $z_0\in K$ we have by (2.1) that

$$|f_n^{(k)}(z_0)-f^{(k)}(z_0)|\leq\frac{k!}{r^k}\max_{z\in\overline{B_r(z_0)}}|f_n(z)-f(z)|\leq\frac{k!}{r^k}\|f_n-f\|_E.$$

Thus, we conclude

$$\|f_n^{(k)}-f^{(k)}\|_K\to 0.$$

□

In the context of complex analysis it is common to use the following synonym for pre-compact subsets of $C(\Omega,\mathbb{C})$.

Definition 2.23. *Let Ω be a region and $\mathscr{F} \subset C(\Omega, \mathbb{C})$. Then $\mathscr{F}$ is called normal, if $\overline{\mathscr{F}}$ is compact in $C(\Omega, \mathbb{C})$.*

Since $C(\Omega, X)$ is a complete metric space, for a family $\mathscr{F} \subset C(\Omega, X)$ the following are equivalent:

(i) $\mathscr{F}$ is normal;
(ii) $\mathscr{F}$ is totally bounded, i.e., for every ϵ there exists finitely many balls with radius ϵ which cover $\mathscr{F}$;
(iii) Every sequence in $\mathscr{F}$ has a convergent subsequence;

Definition 2.24. *A family $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is called locally bounded, if for every $w \in \Omega$ there exists an open neighborhood U_w of w in Ω such that*

$$\sup\{|f(z)| : z \in U_w, f \in \mathscr{F}\} < \infty.$$

Using the usual compactness arguments it is not hard to see that $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is locally bounded if and only if $\sup\{|f(z)| : z \in K, f \in \mathscr{F}\} < \infty$ for every $K \subset \Omega$ compact.

Theorem 2.25 (Montel). *Let $\mathscr{F} \subset \mathrm{Hol}(\Omega)$. Then $\mathscr{F}$ is normal if and only if $\mathscr{F}$ is locally bounded.*

Proof. Assume $\mathscr{F}$ is normal and fix $K \subset \Omega$ compact. Since the map $f \mapsto f|_K$ is continuous, the set $\{f|_K : \quad f \in \overline{\mathscr{F}}\}$ is compact and thus bounded in $(C(K, \mathbb{C}), \|\cdot\|_K)$. This implies that $\mathscr{F}$ is locally bounded.

Assume now that $\mathscr{F}$ is locally bounded. We check the assumptions of the Theorem of Arzela–Ascolu. If $\mathscr{F}$ is locally bounded, then for each $z \in \Omega$, $\{f(z) : \ f \in \mathscr{F}\}$ is bounded in $\mathbb{C}$. Thus, it remains to show that the family is equicontinuous. Fix $z_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$. Using

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} = \frac{z - z_0}{(\zeta - z)(\zeta - z_0)}$$

we obtain for $|z - z_0| < \frac{r}{2}$

$$|f(z) - f(z_0)| \leq \frac{1}{2\pi} \int_{|\zeta - z_0| = r} \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right| d\zeta$$
$$\frac{1}{2\pi} \max_{|\zeta - z_0| = r} |f(\zeta)||z - z_0| \frac{2}{r^2} 2\pi r.$$

Thus, with $M = \max_{|\zeta - z_0| = r}\{|f(\zeta)| : |\zeta - z_0| = r, f \in \mathscr{F}\}$ we obtain that for all $z \in B_{r/2}(z_0)$ and all $f \in \mathscr{F}$

$$|f(z) - f(z_0)| \leq \frac{2}{r} M |z - z_0|$$

and we conclude equicontinuity and thus $\mathscr{F}$ is a normal family. □

Corollary 2.26. *A family $\mathscr{F}$ is compact if and only if $\mathscr{F}$ is closed and (locally) bounded.*

Theorem 2.27 (Vitali's theorem). *Let $(f_n)_{n \in \mathbb{N}}$ is a sequence of holomorphic functions in a region Ω. Suppose that $(f_n)_{n \in \mathbb{N}}$ is locally bounded and that there exists a set D which accumulates on Ω, such that $(f_n)_{n \in \mathbb{N}}$ converges pointwise on D. Then $(f_n)_{n \in \mathbb{N}}$ is a convergent sequence with respect to d.*

Proof. The proof of this theorem is a direct consequence of Montel's theorem, together with a standard but very usefull application of compactness:

Letf_{n_k} be a subsequence converging to $f \in \text{Hol}(\Omega)$, which exists by Montel's theorem. On the other hand, for any $w \in D$ we have

$$f(w) = \lim_{n\to\infty} f_n(w) = \lim_{k\to\infty} f_{n_k}(w).$$

Thus, by the identity theorem, f must be the same for every convergent subsequence and we conclude that $f_n \to f$ in $\text{Hol}(\Omega)$. □

One can also say something about the zeros of a limit of sequence of analytic functions. The following notation will be convenient

$$\text{ord}(f, z_0) = \min\{k \in \mathbb{N}_0 | \mid f^{(k)}(z_0) \neq 0\}.$$

We will first prove the argument principle.

Theorem 2.28. *Let $f \in \text{Hol}(\Omega), f \not\equiv 0$ and assume that z_0, r are so that $\overline{B_r(z_0)} \subset \Omega$ and $f(z) \neq 0$ for $|z - z_0| = r$. Then*

$$N(f, B_r(z_0)) := \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'(z)}{f(z)} dz = \sum_{z \in B_r(z_0)} \text{ord}(f, z_0).$$

Proof. Since $f \not\equiv 0$, f has only finitely many zeros in $B_r(z_0)$. Let us write

$$f(z) = \prod_{k=1}^{n} (z - \alpha_k) g(z),$$

where α_k are the zeros of f in $B_r(z_0)$ counted with multiplicity and g has no zero in $B_r(z_0)$. Then it follows that

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z - \alpha_k} + \frac{g'(z)}{g(z)}.$$

We obtain from Theorem 2.2 and Theorem 2.3 that

$$\int_{|z-z_0|=r} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} \int_{|z-z_0|=r} \frac{1}{z - \alpha_k} dz + \int_{|z-z_0|=r} \frac{g'(z)}{g(z)} dz = 2\pi i n.$$

□

Theorem 2.29. *Let $f_n, f \in \text{Hol}(\Omega)$ and $f_n \to f$ in $\text{Hol}(\Omega)$ and $f \not\equiv 0$. Choose r, z_0 so that $\overline{B_r(z_0)} \subset \Omega$ and $f(z) \neq 0$ for $|z - z_0| = r$. Then, there exists n_0 so that for $n \geq n_0$ we have*

$$N(f_n, B_r(z_0)) = N(f, B_r(z_0)).$$

Proof. Since $f_n \to f$ uniformly on compacts, f_n does not vanish on $\partial B_r(z_0)$ for n sufficiently large. Thus, using that also $f_n' \to f'$ uniformly on compacts, we obtain

$$N(f_n, B_r(z_0)) \to N(f, B_r(z_0)).$$

Since by Theorem 2.28 $N(f_n, B_r(z_0)), N(f, B_r(z_0)) \in \mathbb{N}$ it follows that $N(f_n, B_r(z_0))$ is eventually constant. This concludes the proof. □

We obtain the following Proposition for products of analytic functions:

Proposition 2.30. *Let Ω be a domain and $f_n \in \mathrm{Hol}(\Omega)$ with $|f_n(z)| \le 1$ for all $z \in \Omega$. Define*

$$F_n(z) = \prod_{j=1}^{n} f_j(z)$$

and assume that there is $z_0 \in \Omega$ so that $F_n(z_0)$ has a nonzero limit. Then $(F_n)_{n=1}^{\infty}$ converges to some $F \in \mathrm{Hol}(\Omega)$ and $F \not\equiv 0$. Moreover, for any $z_0 \in \Omega$

$$\operatorname{ord}(F, z_0) = \sum_{j=1}^{\infty} \operatorname{ord}(f_j, z_0). \tag{2.5}$$

Proof. Note that the family $\{F_n \mid n \ge 1\}$ is uniformly bounded by 1 in Ω and hence by Montel's theorem a normal family. As in the proof of Vitali's theorem, it suffices to show that any convergent subsequence has the same limit. Assume that $F_{n_1(n)}, F_{m_1(n)}$ are two convergent subsequences and $F_{n_1(n)} \to F$ and $F_{m_1(n)} \to G$. Note that for any $n \in \mathbb{N}$ and $z \in \mathbb{D}$ $|F_{n+1}(z)| \le |F_n(z)|$. Let n_2 be a further subsequence, so that for any $n \in \mathbb{N}$, $n_2(n_1(n)) \ge m_1(n)$. Thus, it follows for any $z \in \Omega$

$$|F(z)| = \lim_{n\to\infty} |F_{n_2(n_1(n))}| \le \lim_{n\to\infty} |F_{m_1(n)}(z)| = G(z).$$

Reversing the roles of F and G shows that $|F| = |G|$ and thus, $F = cG$ for some unimodular constant c. To see this, take a closed disc B in $\mathbb{D}$, where F and G do not vanish and consider $H = F/G$. Then H is analytic and $|H| = 1$. Thus, by the maximum principle, H is constant. It follows that $F = cG$ on B and by the identity principle, this also holds on Ω. Since $F_n(z_0)$ converges, we conclude that $c = 1$.

It remains to show (2.5). If $F(z_0) \ne 0$, then $f_n(z_0) \ne 0$ for all n and thus the statement follows. Assume now that $F(z_0) = 0$. Since $F \not\equiv 0$, we find $r > 0$ such that $F(z) \ne 0$ on $\overline{B_r}(z_0) \setminus \{z_0\}$. By the same reasoning, we see that $f_j(z) \ne 0$ for $j \ge 1$ and $z \in \overline{B_r}(z_0) \setminus \{z_0\}$. Hence, in particular

$$N(F_n, B_r(z_0)) = \operatorname{ord}(F_n, z_0).$$

From the proof of the argument principle, it follows that

$$N(F_n, B_r(z_0))$$

is eventually constant and equals $\operatorname{ord}(F, z_0)$. Thus, for n sufficiently large

$$\operatorname{ord}(F, z_0) = \operatorname{ord}(F_n, z_0) = \sum_{j=1}^{n} \operatorname{ord}(f_j, z_0) = \sum_{j=1}^{\infty} \operatorname{ord}(f_j, z_0).$$

□

Let us from now on concentrate on the domain $\Omega = \mathbb{D}$. Recall that

$$\mathrm{Aut}(\mathbb{D}) = \{c\tilde{b}_{z_0} \mid z_0 \in \mathbb{D}\}, \quad \tilde{b}_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Note that $|\tilde{b}_{z_0}| < 1$ on $\mathbb{D}$. In the following we will form products of such functions. Thus, it will be convenient, to normalize them to guarantee convergence for at $z = 0$. We thus define

$$b_{z_0}(z) := \begin{cases} -\frac{\overline{z_0}}{|z_0|} \frac{z - z_0}{1 - \overline{z_0} z} & \text{if} \quad z_0 \ne 0 \\ z & \text{if} \quad z_0 = 0. \end{cases}$$

These functions are usually called Blaschke factors, and their products (2.7) are called Blaschke products. They will play an important role in what follows. We will use them to prove a factorization theorem for certain analytic functions in the disc, and those factors will correspond to the zeros if those functions. We will find an explicit condtion, such that Blaschke products converge to a non-trivial function. Note that for $z_0 \neq 0$

$$b_{z_0}(z) = -\frac{\overline{z_0}}{|z_0|}\frac{-z_0}{1} = |z_0| > 0.$$

We need a small lemma:

Lemma 2.31. *Let $0 \leq a_n < 1$. Then*

$$\prod_{j=1}^{\infty}(1-a_j) > 0 \iff \sum_{j=1}^{\infty} a_j < \infty.$$

Proof. Assume that $\sum_{j=1}^{\infty} a_j = \infty$ and note that $\log(1-x) \leq -x$ for $x < 1$. Then, it follows that

$$\prod_{j=1}^{n}(1-a_j) = \exp\left(\sum_{j=1}^{n}\log(1-a_j)\right) \leq \exp\left(-\sum_{j=1}^{n} a_j\right) \to 0.$$

On the other hand, assume that $C = \sum_{j=1}^{\infty} a_j < \infty$. Then there exists j_0 such that for $j \geq j_0$ $a_j \leq 1/2$. For $0 \leq x \leq 1/2$ we have $\log(1-x) \geq -2x$. Thus,

$$\prod_{j=j_0}^{n}(1-a_j) = \exp\left(\sum_{j=j_0}^{n}\log(1-a_j)\right) \geq \exp\left(-2\sum_{j=j_0}^{n} a_j\right) \geq \exp(-2C) > 0.$$

$\square$

If $z, z_0 \in \mathbb{C}$ with $\overline{z_0}z \neq 1$, then

$$1 - \left|\frac{z-z_0}{1-\overline{z_0}z}\right|^2 = \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\overline{z_0}z|^2}. \tag{2.6}$$

Theorem 2.32. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{D}$ and*

$$B(z) = \prod_{j=1}^{\infty} b_{z_j}(z). \tag{2.7}$$

Then

$$B(z) \not\equiv 0 \iff \sum_{j=1}^{\infty}(1-|z_j|) < \infty.$$

If $B(z) \not\equiv 0$, then

$$\operatorname{ord}(B, z_0) = \#\{n \mid z_n = z_0, n \geq 1\}.$$

Proof. Assume that $\sum_{j=1}^{\infty}(1-|z_j|) = \infty$. By precompactness, it suffices to evaluate the pointwise limit. Fix $z \in \mathbb{D}$. Using (2.6), we conclude

$$\prod_{j=1}^{n}|b_j(z)|^2 = \prod_{j=1}^{n}|\frac{z-z_j}{1-\overline{z_j}z}|^2$$

$$= \exp\left(\sum_{j=1}^{n} \log\left|\frac{z - z_j}{1 - \overline{z_j} z}\right|^2\right)$$
$$\leq \exp\left(-\sum_{j=1}^{n}\left(1 - \left|\frac{z - z_j}{1 - \overline{z_j} z}\right|^2\right)\right)$$
$$= \exp\left(-\sum_{j=1}^{n} \frac{(1 - |z|^2)(1 - |z_j|^2)}{|1 - \overline{z_j} z|^2}\right)$$
$$\leq \exp\left(-\frac{1 - |z|^2}{(1 + |z|)^2} \sum_{j=1}^{n}(1 - |z_j|^2)\right) \to 0$$

as $n \to \infty$.

Now assume that $\sum_{j=1}^{\infty}(1 - |z_j|) < \infty$. This implies in particular, that there are only finitely many m so that $z_m = 0$. Thus, we can assume without loss of generality that $z_m \neq 0$ for all $m \geq 1$. By Proposition 2.30 suffices to show that

$$B_n(0) = \prod_{j=1}^{\infty} b_{z_j}(0) = \prod_{j=1}^{n} |z_j|$$

has a nonzero limit. From Lemma 2.31 ir follows that $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$ implies $\prod |z_j| > 0$. The statement about the zeros follows directly from (2.5). □

The condition

$$\sum_{j=1}^{\infty}(1 - |z_j|) < \infty$$

is usually called Blaschke condition.

CHAPTER 3

Harmonic and subharmonic functions

3.1. Defintion and basic properties

DEFINITION 3.1. *Let $\Omega \subset \mathbb{C}$ be open. For a function $u : \Omega \to \mathbb{C}$ which is twice continuously differentiable in $x = \operatorname{Re} z, y = \operatorname{Im} z$, we define*

$$(\Delta u)(z) := ((\partial_x^2 + \partial_y^2)u)(z).$$

The function u is called harmonic in Ω, if for all z in Ω

$$(\Delta u)(z) = 0.$$

EXAMPLE 3.2. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then the Cauchy-Riemann equations imply that $f, \operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic. Write $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, then we have

$$(\Delta u)(z) = \partial_x^2 u(z) + \partial_y^2 u(z) = \partial_x \partial_y v(z) - \partial_y \partial_x v(z) = 0$$
$$(\Delta v)(z) = \partial_x^2 v(z) + \partial_y^2 v(z) = -\partial_x \partial_y u(z) + \partial_y \partial_x u(z) = 0.$$

Interchanging the order of differentiation is justified, since f is $C^\infty(\mathbb{C})$. Since $f = u + iv$ it follows, that also f is harmonic. ◀

Before we proceed with the next theorem, let us recall that due to the Cauchy Riemann equation, one has for a complex differentiable f and $u = \operatorname{Re} f, v = \operatorname{Im} f$

$$\partial_z f(z) = \partial_x u(z) + i\partial_x v(z) = \partial_x u(z) - i\partial_y u(z).$$

PROPOSITION 3.3. *Let Ω be open and convex and $u : \Omega \to \mathbb{R}$ harmonic. Then there exisis $f \in \operatorname{Hol}(\Omega)$, such that $u = \operatorname{Re} f$. Moreover, f is unique up to adding a purely imaginary constant.*

PROOF. Let a harmonic u be given and define

$$g = \partial_x u - i\partial_y u.$$

Then g is (as a function from $\mathbb{R}^2 \to \mathbb{R}^2$) continuously differentiable and satisfies the Cauchy-Riemann equations

$$\partial_x \operatorname{Re} g = \partial_x^2 u = -\partial_y^2 u = \partial_y \operatorname{Im} g$$
$$\partial_x \operatorname{Im} g = -\partial_x \partial_y u = -\partial_y \partial_x u = \partial_y \operatorname{Re} g$$

Thus, $g \in \operatorname{Hol}(\Omega)$ and since Ω is convex, there exists $f \in \operatorname{Hol}(\Omega)$ with $f' = g$ and $f(z_0) = u(z_0)$ for some $z_0 \in \Omega$. It follows that by construction

$$\partial_x(\operatorname{Re} f - u) = \partial_y(\operatorname{Re} f - u) = 0$$

and thus $\operatorname{Re} f - u$ must be a constant. By evaluation at z_0 it follows that $\operatorname{Re} f = u$. Take any other $\tilde{f} \in \operatorname{Hol}(\Omega)$, with $\operatorname{Re} \tilde{f} = u$. Then,

$$(f - \tilde{f})' = f' - \tilde{f}' = \partial_x u - i\partial_y u - (\partial_x \operatorname{Re} \tilde{f} - i\partial_y \operatorname{Re} \tilde{f}) = 0$$

and thus, $f - \tilde{f}$ must be constant. Since $\operatorname{Re} f(z_0) = \operatorname{Re} \tilde{f}(z_0)$ this constant must be purely imaginary, and the uniqueness claim follows. □

We obtain immediately the following corollary:

Corollary 3.4. *Let u be harmonic in some open set Ω. Then $u \in C^\infty(\Omega)$.*

Proof. It suffices to show that for $u \in C^\infty(B_r(z_0))$ for any $z_0 \in \Omega$ and some $r > 0$. By considering the $\operatorname{Re} u, \operatorname{Im} u$ separately, we can assume that u is real valued. Fix $z_0 \in \Omega$ and $r > 0$ so that $B_r(z_0) \subset \Omega$. Since this is convex, we find $f \in \operatorname{Hol}(\Omega)$ with $\operatorname{Re} f = u$. Since f is analytic if follows that $f \in C^\infty(B_r(z_0))$ and thus $u \in C^\infty(B_r(z_0))$. □

Another immediate corollary is the following:

Corollary 3.5. *Let Ω_1, Ω_2 be open and $g : \Omega_1 \to \Omega_2$ be holomorphic and u harmonic on Ω_2. Then $u \circ g$ is harmonic on Ω_1.*

Proof. Fix z_1 in Ω_1 and set $z_2 = g(z_1)$. Let $r_2 > 0$ such that $B_{r_2}(z_2) \subset \Omega_2$. On $B_{r_2}(z_2)$ we find $f \in \operatorname{Hol}(B_{r_2}(z_2))$ so that $\operatorname{Re} f = u$. Since f is continuous $U_1 = f^{-1}(B_{r_2}(z_2))$ is open and $z_1 \in U_1$. Thus, we find r_1 so that $B_{r_1}(z_1) \subset U_1$. It follows that $f \circ g \in \operatorname{Hol}(B_{r_1}(z_1))$ and $\operatorname{Re}(f \circ g) = u \circ g$. □

As a direct consequence, we also obtain that harmonic functions satisfy the mean value property:

Theorem 3.6. *Let Ω be harmonic and z_0, r so that $\overline{B_r(z_0)} \subset \Omega$. Then,*

$$u(z_0) = \int_{-\pi}^{\pi} h(z_0 + re^{it}) \frac{dt}{2\pi}.$$

Proof. Again, by taking $\operatorname{Re} u, \operatorname{Im} u$ we can assume that u is real valued. Choose $r' > r$ so that u is harmonic on $B_{r'}(z_0)$. Then we find $f \in \operatorname{Hol}(B_{r'}(z_0))$ so that $\operatorname{Re} f = u$. By Cauchy's integral formula 2.3 and $\gamma(t) = z_0 + re^{it}$ it follows that

$$f(z_0) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z_0 + re^{it})}{z_0 + re^{it} - z_0} rie^{it} dt = \int_{-\pi}^{\pi} f(z_0 + re^{it}) \frac{dt}{2\pi}.$$

Taking the real part on both sides, finishes the proof. □

There also exists an identity principle for harmonic functions:

Theorem 3.7. *Let Ω be a region and u, h harmonic in Ω. If there exists an open set $U \subset \Omega, U \neq \emptyset$ so that $u|_U = h|_U$, then $h = u$ in Ω.*

Proof. Let $v = u - h$ and define $g = \partial_x v - i\partial_y v$. Then $g \in \operatorname{Hol}(\Omega)$ and $g|_U \equiv 0$. Thus, it follows that $g \equiv 0$ on Ω. Hence $\partial_x v = \partial_y v = 0$ and thus v must be constant in Ω. Since $v|_U = 0$ it follows that $v \equiv 0$ and thus $u = h$ in Ω. □

We note that the identity principle is weaker, compared to the one for analytic functions. That U is open was used to conclude that $g|_U = 0$. However, this is not due to the proof but indeed a property of harmonic functions. As an example consider $u = \operatorname{Im} z$ on $\Omega = \mathbb{C}$. Then u is harmonic and $u = 0$ on $\mathbb{R}$, but $u \not\equiv 0$ on Ω.

We conclude this Section with a maximum principle for harmonic functions which we will first prove for real valued harmonic functions.

Theorem 3.8 (Maximum principle). *Let Ω be a region and u a real value harmonic function in Ω. Then:*

(i) if there exists r, z_0 such that for any $z \in B_r(z_0)$ $u(z) \leq u(z_0)$, then u is constant.
(ii) if Ω is bounded and f has a continuous extension to $\overline{\Omega}$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ on Ω.

Proof. (i): We find $f \in \operatorname{Hol}(B_r(z_0))$ with $\operatorname{Re} f = u$. Consider $g = e^f$. Then $|g| = e^{\operatorname{Re} f} = e^u$. Thus, $|g|$ attains a local maximum at z_0 and it follows from the Maximum modulus principle Theorem 2.14 that g and therefore also u must be constant.

(ii): This follows in the same way as Theorem 2.14 (ii). □

In contrast to complex valued holomorphic functions, we don't have to take the modulus. However, it is not hard to also show a maximum modulus principle for complex valued harmonic functions.

3.2. The Dirichlet problem on $\mathbb{D}$

This section will be concerned with the Dirichlet problem for the Laplace equation on the unit disc. Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then for $g \in C(\mathbb{T}, \mathbb{C})$ we look for a solution of

$$\begin{aligned} \Delta u(z) &= 0 \quad \forall z \in \mathbb{D}, \\ \lim_{z \to \zeta} u(z) &= g(\zeta) \ \forall \zeta \in \mathbb{D}. \end{aligned} \tag{3.1}$$

Let us start with a uniqueness statement about a classical solution to this problem.

Proposition 3.9. *There exists at most one solution of (3.1).*

Proof. It follows from the maximum principle that if g is real valued, then so is u (in case it exists). Thus, by taking real and imaginary parts, it suffices to consider real valued u and g. Assume that u_1, u_2 are two solutions and consider $v = u_1 - u_2$. Then v is harmonic in $\mathbb{D}$, extends continuously to $\overline{\mathbb{D}}$ and $v|_{\mathbb{T}} \equiv 0$. Thus, $v \equiv 0$. □

To show existence, we will introduce the Poisson kernel:

Definition 3.10. *On $\mathbb{D} \times \mathbb{T}$, we define the Poisson kernel:*

$$P(z, \zeta) := \operatorname{Re} \frac{\zeta + z}{\zeta - z} = \frac{1 - |z|^2}{|\zeta - z|^2}.$$

Let μ be a complex (i.e., in particular finite) Borel measure on $\mathbb{T}$, then we define the Poisson integral of μ by

$$\mathscr{P}[d\mu](z) = \int_{\mathbb{T}} P(z, \zeta) d\mu(\zeta).$$

Proposition 3.11. *Let μ be a complex measure. Then*

$$\int \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \tag{3.2}$$

defines an analytic and $\mathscr{P}[\mu]$ a harmonic function in $\mathbb{D}$.

Proof. Let first μ be a real (finite) measure. We have

$$1 + 2\sum_{j=1}^{\infty} \left(\frac{z}{\zeta}\right)^j = 1 + 2\frac{\frac{z}{\zeta}}{1 - \frac{z}{\zeta}} = \frac{\zeta + z}{\zeta - z}$$

The series

$$\sum_{j=1}^{\infty} \left(\frac{z}{\zeta}\right)^j$$

converges uniformly for $\zeta \in \mathbb{T}$. Thus we can interchange summation and integration and get

$$\int \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = \mu(\mathbb{T}) + 2\sum_{j=1}^{\infty} \zeta^{-j} d\mu(\zeta) z^j.$$

Thus, (3.2) defines an analytic function and $\mathscr{P}[\mu]$ as the real part of it a harmonic function. Since analyticity and harmonicity is preserved under taking finite linear combinations (with constant coefficients) we get the claim for complex measures. □

Remark 3.12. *(i) If $z = re^{i\theta}$ and $\zeta = e^{it}$, then*

$$\begin{aligned} \frac{1 - |z|^2}{|\zeta - z|^2} &= \operatorname{Re} \frac{1 - r^2}{|1 - re^{i(\theta - t)}|^2} = \frac{1 - r^2}{(1 - r\cos(\theta - t))^2 + r^2 \sin(\theta - t)^2} \\ &= \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \end{aligned} \tag{3.3}$$

Often, the Poisson kernel is also introduced in this way.

(ii) Let λ denote the normalized Lebesgue measure on $\mathbb{T}$. That is, if $\frac{dt}{2\pi}|_{[-\pi,\pi)}$ denotes the Lebesgue measure on $[-\pi, \pi)$ and

$$\psi : \begin{cases} [-\pi, \pi) & \to & \mathbb{T} \\ t & \mapsto & e^{it} \end{cases},$$

then λ corresponds to the pushforward measure of $\frac{dt}{2\pi}|_{[-\pi,\pi)}$ under ψ. If μ is absolutely continuous, i.e., if $d\mu = f d\lambda$ for some $f \in L^1(\mathbb{T})$, then we also write

$$\mathscr{P}[d\mu](z) = \mathscr{P}[f](z).$$

Lemma 3.13. *The Poisson kernel has the properties:*

(i) For $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, it holds that $P(z, \zeta) > 0$;
(ii) For $z \in \mathbb{D}$ it holds that $\int_{-\pi}^{\pi} P(z, e^{it}) \frac{dt}{2\pi} = 1$;
(iii) For every $\delta > 0$ and $\zeta_0 \in \mathbb{T}$ it holds that

$$\lim_{z \to \zeta_0} \sup_{|\zeta - \zeta_0| > \delta} P(z, \zeta) = 0.$$

PROOF. (i): This follows directly from the definition.
(ii): By Cauchy's integral formula, Theorem (2.3), we obtain

$$\int_{-\pi}^{\pi} P(z, e^{it}) \frac{dt}{2\pi} = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}\right) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{-\pi}^{\pi}\left(\frac{2}{\zeta - z} - \frac{1}{\zeta}\right)d\zeta\right) = \operatorname{Re}(2-1) = 1.$$

(iii): If $|z-\zeta_0| < \delta$ and $|\zeta - \zeta_0| \geq \delta$, then

$$|\zeta - z|^2 = |\zeta - \zeta_0 + \zeta_0 - z|^2 \geq (\delta - |z - \zeta_0|)^2$$

and thus,

$$\sup_{|\zeta-\zeta_0|\geq\delta} P(z,\zeta) \leq \frac{1-|z|^2}{(\delta - |z-\zeta_0|)^2} \to 0$$

as $z \to \zeta_0$. □

We are now ready to prove existence for the Dirichlet problem:

THEOREM 3.14. *Let $f \in C(\mathbb{T})$ and define*

$$F(z) = \begin{cases} f(z), & z \in \mathbb{T}, \\ \mathscr{P}[f](z), & z \in \mathbb{D}. \end{cases}$$

Then we have that F is harmonic in $\mathbb{D}$ and extends continuously to the boundary. On the other hand, if u is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, then on $\mathbb{D}$ we have

$$u(z) = \mathscr{P}[u](z).$$

PROOF. Since $f d\lambda$ is a complex measure, we obtain from Proposition 3.11 that F is harmonic. Fix $z_0 \in \mathbb{D}$ and $\epsilon > 0$. Choose $\delta > 0$, such that for $|\zeta - \zeta_0| \leq \delta$

$$|f(\zeta) - f(\zeta_0)| \leq \frac{\epsilon}{2}$$

Moreover, choose ϵ_1 such that for $|z - \zeta_0| < \delta_1$ we have that

$$\sup_{|\zeta-\zeta_0|>\delta} P(z,\zeta) < \frac{\epsilon}{4\|f\|_{[-\pi,\pi]}}.$$

With this we have

$$\begin{aligned} &\left|\int_{\mathbb{T}} P(z,\zeta) f(\zeta) d\lambda(\zeta) - f(\zeta_0)\right| \\ &\leq \int_{\mathbb{T}} |P(z,\zeta)f(\zeta) - P(z,\zeta)f(\zeta_0)|\, d\lambda(\zeta) \\ &= \int_{\mathbb{T}\cap|\zeta-\zeta_0|\leq\delta} P(z,\zeta)\,|f(\zeta) - f(\zeta_0)|\, d\lambda(\zeta) + \int_{\mathbb{T}\cap|\zeta-\zeta_0|>\delta} P(z,\zeta)\,|f(\zeta) - f(\zeta_0)|\, d\lambda(\zeta) \\ &< \frac{\epsilon}{2} + 2\|f\|_{[-\pi,\pi]} \frac{\epsilon}{4\|f\|_{[-\pi,\pi]}} = \epsilon. \end{aligned}$$

On the other hand if u is as in the Theorem, then the claim follows from the uniqueness of the Dirichlet problem Proposition (3.9). $\square$

REMARK 3.15. *Let $z_0 \in \mathbb{D}$ and $r > 0$ and define*

$$\psi := \begin{cases} \overline{B_r(z_0)} & \to & \mathbb{D} \\ w & \mapsto & \frac{w-z_0}{r} \end{cases}$$

Then for any complex measure on $\partial B_r(z_0) = \{w \mid |w - z_0| = r\}$ we have

$$\int_{\mathbb{T}} P(z,\zeta) d\mu^{\psi}(\zeta) = \int_{\partial B_r(z_0)} P(z,\psi(s)) d\mu(s).$$

Then it follows from Corollary 3.5 that

$$\mathscr{P}_{r,z_0}[\mu](w) := \int_{\partial B_r(z_0)} P(\psi(w),\psi(s)) d\mu(s)$$

inherits all properties from the Poisson integral on $\mathbb{T}$. A direct computation shows that

$$P(\psi(w),\psi(\zeta)) = \frac{r^2 - |w - z_0|^2}{|w-\zeta|^2}.$$

As a corollary we obtain the so-called Poisson Jensen formula:

COROLLARY 3.16 (Poisson-Jensen formula). *Assume that u is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then there exists a real constant c such that for $z \in \mathbb{D}$*

$$u(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} u(e^{it}) \frac{dt}{2\pi} + ic.$$

PROOF. The function $\operatorname{Re} u(z)$ is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Thus

$$\operatorname{Re} u(z) = \mathscr{P}[\operatorname{Re} u](z).$$

We have that $\mathscr{P}[\operatorname{Re} u](z)$ is the real part of

$$\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} u(e^{it}) \frac{dt}{2\pi}$$

which is analytic in $\mathbb{D}$ due to Proposition 3.11. Due to Theorem (3.3)

$$u(z) - \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} u(e^{it}) \frac{dt}{2\pi}$$

must be a constant. Since the real parts coincide, this constant must be purely imaginary. $\square$

3.3. Properties of harmonic functions

We have already seen in Theorem 3.6 that harmonic functions satisfy a mean-value property. We will now see that this property actually characterizes harmonic functions in the following sense.

THEOREM 3.17. *Assume that u is continuous on an open set Ω. And assume that u satisfies the local mean-value property. That is, for every $w \in \Omega$, there exists $R > 0$ such that for every $0 < r < R$ it holds that*

$$u(w) = \int_{-\pi}^{\pi} u(w + re^{it}) \frac{dt}{2\pi}.$$

Then, u is harmonic in Ω.

PROOF. Again, we can assume without loss of generality that u is real valued. It is enough to show that u is harmonic in every disc $B_r(z_0)$ with $\overline{B_r(z_0)} \subset \Omega$. Fix such $B_r(z_0)$, and define $h : B_r(z_0) \to \mathbb{R}$ by

$$h(z) = \begin{cases} u(z) - \mathscr{P}_{r,z_0}[u](z) & z \in B_r(z_0) \\ 0 & |z - z_0| = r. \end{cases}$$

Then h is continuous on $\overline{B_r(z_0)}$ and has the local mean value property on $B_r(z_0)$. Since $\overline{B_r(z_0)}$, h attains its maximum on $\overline{B_r(z_0)}$. Set $M = \max_{|z-z_0| \leq r} h(z)$. Define

$$A = \{z \in B_r(z_0) \mid h(z) < M\}, \quad B = \{z \in B_r(z_0) \mid h(z) = M\}.$$

A is open by continuity of h. Also B is open, because if $w \in B$, then the local mean value property implies that $u|_{B_\rho(w)} \equiv M$, for some sufficiently small ρ. Since $B_r(z_0)$ is connected, we conclude that either $B_r(z_0) = A$ or $B_r(z_0) = B$. In the first case, h attains its maximum on the boundary, and we conclude that $h \leq 0$ on $\overline{B_r(z_0)}$. In the second case $h \equiv M$ on $B_r(z_0)$ and we conclude that $M = 0$. In any case, we obtain that $h \leq 0$ on $\overline{B_r(z_0)}$. A similar argument shows that $h \geq 0$ and thus $h = 0$. That is $u = P_{r,z_0}[u](z)$ on $B_r(z_0)$ which is harmonic. This concludes the proof. □

The following property could also be easily derived from Proposition 3.3.

COROLLARY 3.18. *Let Ω be a region and u_n harmonic in Ω and for some $u : \Omega \to \mathbb{C}$, we have that $u_n \to u$ uniformly on compact subsets of Ω. Then u is harmonic.*

PROOF. The local mean value property is preserved, when taking uniform limits. □

LEMMA 3.19 (Harnack's inequality). *Let u a non-negative harmonic function on $B_R(z_0)$. Then for $r < R$ and $-\pi \leq \theta < \pi$ it holds that*

$$\frac{R-r}{R+r} h(z_0) \leq h(z_0 + re^{i\theta}) \leq \frac{R+r}{R-r} h(z_0).$$

PROOF. We write $\zeta = z_0 + Re^{it}$ and $z = z_0 + re^{i\theta}$. Then we get for the Poisson kernel that

$$\frac{R^2 - |z - z_0|^2}{|z - \zeta|^2} = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t - \theta)}.$$

From $(R - r)^2 \leq R^2 + r^2 - 2Rr\cos(t - \theta) \leq (R + r)^2$ it follows that

$$\frac{R-r}{R+r} \leq \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t - \theta)} \leq \frac{R+r}{R-r}.$$

Chose $\rho < R$. Then is follows from the Poisson integration formula and the mean value property for $r < \rho$

$$h(z_0 + re^{i\theta}) = \int_\pi^\pi \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(t-\theta)} h(z_0 + \rho e^{it}) \frac{dt}{2\pi}$$
$$\leq \frac{\rho + r}{\rho - r} \int_\pi^\pi h(z_0 + \rho e^{it}) \frac{dt}{2\pi} = \frac{\rho + r}{\rho - r} h(z_0).$$

Sending $\rho \to R$ proves the upper estimate. The lower estimate works in the same way. □

We get the following precompactness criteria for positive harmonic functions:

THEOREM 3.20 (Harnack's principle). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real valued harmonic functions on a domain Ω and assume that $u_1 \leq u_2 \leq \dots$. Then either $\lim_{n\to\infty} u_n = \infty$ uniformly on compact subsets of Ω or, there exists a harmonic function u such that*

$$\lim_{n\to\infty} u_n = u.$$

PROOF. By considering $\tilde{u}_n = u_n - u_1$ we can assume that $u_1 = 0$. Define pointwise $u(z) = \lim_{n\in\mathbb{N}} u_n(z)$ and define

$$A = \{z \in \Omega \mid u(z) = \infty\}, \quad B = \{z \in \Omega \mid u(z) < \infty\}.$$

Harnack's inequality implies that A and B are open and since Ω is connected, either $\Omega = A$ or $\Omega = B$. In the first case, again by Harnack's inequality, we see that u_n tends to ∞ locally uniformly and thus uniformly on compacts.

If $\Omega = B$. Then we conclude that pointwise $u_n(z)$ is a Cauchy sequence. Since for $m \geq n$, Harnack's inequality can be applied to $u_m - u_n$, we get that u_n is a Cauchy sequence in the local uniform topology. We conclude that $u_n \to u$ uniformly on compacts. By Corollary 3.18 □

3.4. Boundary behaviour of Poisson integrals

If $f \in C(\mathbb{T})$, we have seen that

$$\lim_{r\to 1} \mathscr{P}[f](re^{i\theta}) = f(e^{i\theta}).$$

We are interested in this type of boundary behavior for arbitrary Poisson integrals $\mathscr{P}[\mu]$.

If μ is a complex measure on $\mathbb{R}$, we define its distribution function by

$$F_\mu(x) = \mu((-\infty, x]), \quad x \in (a, b].$$

The function F_μ is of bounded variation. We collect some properties of functions of bounded variation below. Let F be of bounded variation, then:

(i) $\operatorname{Re} F_\mu, \operatorname{Im} F_\mu$ are of bounded variation;
(ii) If $F : \mathbb{R} \to \mathbb{R}$, then F can be expressed as the difference of two bounded increasing functions;
(iii) F' exists a.e. with respect to the Lebesgue measure, (Lebesgue differentiation theorem!) and $F' \in L^1_{\text{loc}}(\mathbb{R}, dx)$;
(iv) If μ is a complex measure, then there exists a complex measure μ_s, such that $d\mu(x) = F'_\mu(x)dx + d\mu_s(x)$;
(v) $F(x+) = \lim_{s\searrow x} F(s)$ and $F(x-) = \lim_{s\swarrow x} F(s)$ exists for every $x \in \mathbb{R}$;

We will need the following integration by parts rule.

LEMMA 3.21. *Let $-\infty < a < b < \infty$. μ, ν be complex measures on $\mathbb{R}$ and F_μ, F_ν as above. If either μ or ν have no point masses on $[a, b)$, then*

$$\int_{[a,b)} F_\nu d\mu = F_\nu(b)F_\mu(b) - F_\nu(a)F_\mu(a) - \int_{[a,b)} F_\mu d\nu.$$

PROOF. Since F_μ, F_ν are differences of increasing functions, a simple calculation shows that it suffices to consider F_μ, F_ν non-negative and increasing. For definiteness assume that μ has no point masses.

By Fubini applied to the set $\Omega = \{(x, y) \mid a < x \leq y \leq b\}$ we get

$$\begin{aligned}(\mu \otimes \nu)(\Omega) = \int_{(a,b]} \nu((a, y])d\mu(x) &= \int_{(a,b]} (F_\nu(y) - F_\nu(a))d\mu(x)\\ &= \int_{(a,b]} F_\nu d\mu - F_\nu(a)(F_\mu(b) - F_\mu(a))\end{aligned}$$

On the other hand, we get

$$\begin{aligned}(\mu \otimes \nu)(\Omega) = \int_{(a,b]} \mu([x, b])d\nu(x) = \int_{(a,b]} \mu((x, b])d\nu(x) &= \int_{(a,b]} (F_\mu(b) - F_\mu(x))d\nu(x)\\ &= F_\mu(b)(F_\nu(b) - F_\nu(a)) - \int_{(a,b]} F_\mu d\nu.\end{aligned}$$

□

Recall the map

$$\psi : \begin{cases} (-\pi, \pi] & \to & \mathbb{T} \\ t & \mapsto & e^{it} \end{cases},$$

and let $\phi = \psi^{-1}$. Then every complex measure μ induces a measure on $(-\pi, \pi]$ by μ^ϕ. For $\theta_0 \in (-\pi, \pi)$, we define (if it exists) by

$$(D\mu)(e^{i\theta_0}) = F'_{\mu^\phi}(\theta_0).$$

Clearly, by choosing another branch of the argument, we can also allow $\theta_0 = -\pi$.

THEOREM 3.22 (Fatou). *Let μ be a complex measure on $\mathbb{T}$ and assume that for $\theta_0 \in [-\pi, \pi)$ $(D\mu)(e^{i\theta_0})$ exists. Then*

$$\lim_{r\to1} \mathscr{P}[\mu](re^{i\theta_0}) = (D\mu)(e^{i\theta_0}).$$

PROOF. By rotating and subtracting a constant, we can assume that $\theta_0 = 0$ and $(D\mu)(1) = 0$. We apply Lemma 3.21 with $d\nu(s) = \partial_s P(r, e^{is})ds$ and $\mu = \mu^\phi$ and note that ν has no point masses and $F_\nu(s) = P(r, e^{is})$.

In the following, o notation is related to $r \to 1$. Note that $P(r, e^{\pm i\pi}) \to 0$ as $r \to 1$ and we get

$$\int_{(-\pi,\pi]} P(r, e^{it})d\mu^\phi(t) = P(r, e^{i\pi})F_{\mu^\phi}(\pi) - P(r, e^{-i\pi})F_{\mu^\phi}(-\pi) - \int_{[-\pi,\pi)} F_{\mu^\phi}(s)\partial_s P(r, e^{is})ds \tag{3.4}$$

$$= -\int_{[-\pi,\pi)} F_{\mu^\phi}(s)\partial_s P(r,e^{is})ds + o(1). \tag{3.5}$$

We have

$$\partial_s P(r,e^{is}) = \partial_s \frac{1-r^2}{1+r^2+2r\cos(s)} = \frac{(1-r^2)(2r\sin(s))}{(1+r^2+2r\cos(s))^2}.$$

Thus, for $s \in [-\pi,\pi] \setminus (-\delta,\delta)$ we see that

$$|\partial_s P(r,e^{is})| \leq \frac{(1-r^2)2r}{((1-r)^2+2r(1-\cos(\delta)))^2}$$

We conclude that

$$\int_{[-\pi,\pi)} P(r,e^{it})d\mu^\phi(t) = \int_{-\delta}^{\delta} F_{\mu^\phi}(s)\partial_s P(r,e^{is})ds + o(1).$$

Note that $-s\partial_s P(r,e^{is}) \geq 0$ and is even. Thus we get

$$\int_{-\delta}^{\delta} F_{\mu^\phi}(s)\partial_s P(r,e^{is})ds = \int_0^{\delta} \left(\frac{F_{\mu^\phi}(s) - F_\mu(-s)}{s}\right) s\partial_s P(r,e^{is})ds.$$

Fix $\epsilon > 0$ and assume that $\delta > 0$ is so that for $s \leq \delta$ it holds that

$$\left(\frac{F_{\mu^\phi}(s) - F_{\mu^\phi}(-s)}{s}\right) < \epsilon.$$

Then we get

$$\begin{aligned}\int_0^{\delta} \left|\frac{F_{\mu^\phi}(s) - F_{\mu^\phi}(-s)}{s}\right| s\partial_s P(r,e^{is})ds &< \epsilon \int_{-\pi}^{\pi} s\partial_s P(r,e^{is})ds \\ &= \epsilon\left(\pi P(r,e^{i\pi}) + \pi P(r,e^{-\pi i}) - \int_{-\pi}^{\pi} P(r,e^{is})ds\right) \\ &\leq \epsilon C.\end{aligned}$$

Since ϵ was arbitrary, this finishes the proof. $\square$

Corollary 3.23. *Let μ be a complex measure on $\mathbb{T}$ and $d\mu(\zeta) = f(\zeta)d\lambda(\zeta) + d\mu_s(\zeta)$ its Lebesgue decomposition into its absolutely continuous and singular continuous part. Then $\mathscr{P}[\mu](\zeta) := \lim_{r\to 1} \mathscr{P}[\mu](r\zeta)$ exists for a.e. $\zeta \in \mathbb{T}$ and $\mathscr{P}[\mu](\zeta) = f(\zeta)$ a.e.*

In the function theory on the unit disc the limit $\lim_{r\to 1} F(r\zeta)$ is called radial limit. It is also common to consider so-called non-tangential limits. For $0 < \alpha < 1$, let $\Omega_0(\alpha)$ be the convex hull of 1 and $D(0,\alpha)$ and $\Omega_{\theta_0}(\alpha) = e^{i\theta_0}\Omega_0(\alpha)$.

Definition 3.24. *Let $f : \mathbb{D} \to \mathbb{C}$. We say that f has nontangential limit at $\zeta_0 = e^{i\theta_0}$, if the following limit exists for any $0 < \alpha < 1$ and does not depend on α:*

$$\lim_{\substack{\zeta\to\zeta_0 \\ \zeta\in\Omega_{\theta_0}(\alpha)}} f(\zeta).$$

In this case we write

$$f^*(\zeta) = \lim_{z \hat{\to} \zeta} f(z).$$

REMARK 3.25. *In general for $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$ it holds that $1 - |z| \leq |\zeta - z|$, but it may happen that $\frac{1-|z|}{|z-\zeta|}$ converges to 0 as $z \to \zeta$. For nontangential limits this cannot happen. That is, for any fixed $0 < \alpha < 1$ one can find γ_ρ so that*

$$\gamma_\alpha \leq \frac{1-|z|}{|z - e^{i\theta_0}|}, \quad z \in \Omega_{\theta_0}(\alpha).$$

A neat application of Montel's theorem shows that for bounded analytic functions on $\mathbb{D}$ these two notions are equivalent:

PROPOSITION 3.26. *Let $f \in \mathrm{Hol}(\mathbb{D})$, f is uniformly bounded on $\mathbb{D}$ and assume that f has radial limits at $e^{i\theta_0}$. Then f has nontangential limits at $e^{i\theta_0}$.*

PROOF. It will be more convenient to work with the disc $B_1(1)$ and consider the limit $z \to 0$, which can be achieved by rotation and translation. Let

$$\Omega_\alpha = \{z \mid |\arg z| < \alpha, \quad |z| < 1/2\}.$$

By assumption

$$\lim_{r \searrow 0} f(r) = L$$

exists. We want to show that that for any $z_n \in \Omega_\alpha$ $z_n \to 0$ it holds that

$$\lim_{n\to\infty} f(z_n) = L.$$

We claim that uniformly on compact subsets of Ω_α it holds that

$$\lim_{r\to 0} f(rz) = L.$$

For $z \in \Omega_\alpha$, define $f_r(z) = f(rz)$. Then by Montel's theorem the family $\{f_r \mid r \in (0,1)\}$ is precompact. Let r_n be so that

$$\lim_{n\to\infty} f_{r_n} = f_\infty.$$

Then, in particular, for $z \in (0, 1/2)$ we have

$$\lim_{n\to\infty} f_{r_n}(z) = \lim_{n\to\infty} f(zr_n) = \lim_{\tilde{r}_n \to 0} f(\tilde{r}_n) = L.$$

Since $(0, 1/2)$ has an accumulation point in Ω_α it follows that $f_\infty = L$. Now take $z_n \in \Omega_{\alpha'}$, with $0 < \alpha' < a$ and $z_n \to 0$. Then we find w_n in some compact subset K of Ω_α and r_n with $r_n \to 0$ such that $z_n = r_n w_n$. Thus, the claim follows by

$$\lim_{n\to\infty} |f(z_n) - L| = \lim_{n\to\infty} |f(r_n w_n) - L| \leq \lim_{n\to\infty} \sup_{w \in K} |f_{r_n}(w) - L| = 0.$$

□

3.5. Representation theorems

Let $f : \mathbb{T} \to \mathbb{C}$ be measurable and define for $0 < p < \infty$

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(\zeta)|^p d\lambda(\zeta) \right)^{1/p}$$

and for $p = \infty$

$$\|f\|_p := \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} |f(\zeta)|.$$

Moreover, let $L^p(\mathbb{T})$ denote the set (of equivalence classes of functions that coincide λ a.e.) of all measurable functions f with $\|f\|_p < \infty$.

For $1 \le p \le \infty$, $L^p(\mathbb{T})$ is a Banach space and for $0 < p < 1$, define $d_p(f,g) = \|f-g\|_p^p$ a metric. Recall that for $1 \le p < \infty$ and q so that $\frac{1}{p} + \frac{1}{q} = 1$ defines

$$f \mapsto \int_{\mathbb{T}} fg d\lambda$$

and Isomorphism from $L^q(\mathbb{T})$ to $L^p(\mathbb{T})^*$.

For $f : \mathbb{D} \to \mathbb{C}$ (f will mainly be assumed to be harmonic) and $0 \le r < 1$ we define a function $f_r : \mathbb{T} \to \mathbb{C}$ by

$$f_r(\zeta) = f(r\zeta).$$

Note that if f is for instance continuous in $\mathbb{D}$, then for any $p \in (0,\infty]$ and $r < 1$, $f_r \in L^p(\mathbb{T})$. A crucial element of Hardy Space theory is, how f behaves when r tends to 1.

THEOREM 3.27. *Let u be a harmonic function in $\mathbb{D}$. Then u is the Poisson integral of*

(i) a complex measure, if and only if $\sup_{r\in[0,1)} \|u_r\|_1 < \infty$;

(ii) a function $f \in L^p$, where $p \in (1,\infty)$, if and only if $\sup_{r\in[0,1)} \|u_r\|_p < \infty$. *In this case* $\lim_{r\nearrow 1} \|u_r - f\|_p = 0$;

(iii) a function $f \in L^\infty(\mathbb{T})$ if and only if $\sup_{r\in[0,1)} \|u_r\|_\infty < \infty$;

(iv) a function $f \in C(\mathbb{T})$ if and only if $\lim_{r\nearrow 1} \|u_r - f\|_\infty = 0$;

(v) a finite non-negative measure, if and only if $u \ge 0$ in $\mathbb{D}$.

In all of the cases the measures and the functions are uniquely determined by u.

PROOF. (i): Assume that $u = \mathscr{P}[\mu]$. Since $P(z,\zeta) > 0$, we get

$$u(z) = \left| \int P(z,\zeta) d\mu(\zeta) \right| \le \int P(z,\zeta) d|\mu(\zeta)| = \mathscr{P}(d|\mu|)(z) := h(z).$$

Since h is harmonic, it satisfies the mean value property. Thus, we get

$$\|u_r\|_1 = \int_{\mathbb{T}} |u(r\zeta)| d\lambda(\zeta) \le \int_{\mathbb{T}} h(r\zeta) d\lambda(\zeta) = h(0)$$

Thus,

$$\sup_{r\in[0,1)} \|u_r\|_1 \le h(0) < \infty.$$

Now we assume that $\sup_{r\in[0,1)} \|u_r\|_1 = M < \infty$. Consider $\mu_r = u_r(\xi) d\lambda$ as elements of $(C(\mathbb{T}), \|\cdot\|_\infty)^*$, then $\|\mu_r\| = \|u_r\|_1$ and thus

$$\sup \|\mu_r\| < \infty$$

By a version of Banach-Alaoglu (Note that $(C(\mathbb{T}), \|\cdot\|_\infty)$ is separable), we find $r_n \to \infty$ and a complex measure μ such that

$$\lim_{n\to\infty} \mu_{r_n} = \mu,$$

in the w^*-topology of $(C(\mathbb{T}), \|\cdot\|_\infty)^*$. The function u_r is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and thus, $u_r = \mathscr{P}[u_r]$. Moreover, since for fixed $z \in \mathbb{D}$, $\mathscr{P}(\cdot, z) \in C(\mathbb{T})$, we get

$$u(z) = \lim_{n\to\infty} u_{r_n}(z)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int P(z,\zeta) u_{r_n}(\zeta) d\lambda(\zeta) \\
&= \lim_{n\to\infty} \int P(z,\zeta) d\mu_{r_n}(\zeta) \\
&= \lim_{n\to\infty} \int P(z,\zeta) d\mu(\zeta) = \mathscr{P}[\mu](z).
\end{aligned}
$$

(ii),(iii) $\Longleftarrow$: Consider u_r as functions in $L^p(\mathbb{T}) = L^q(\mathbb{T})^*$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\|u_r\|_{L^p(\mathbb{T})} = \|u_r\|_{L^q(\mathbb{T})^*}$. If

$$\sup_{r\in[0,1)} \|u_r\|_{L^p(\mathbb{T})} < \infty,$$

then we find $r_n \to 1$ and $f \in L^p$ such that $u_{r_n} \to f$ in the w^*-sense of $(L^q(\mathbb{T}))^*$. By the same argument as above, we see that $u = \mathscr{P}[f]$.

(v): If μ is a positive and finite measure, then $\mathscr{P}[\mu] \geq 0$, since the Poisson kernel is positive. On the other hand, if $u \geq 0$, then it follows from the mean value property, that

$$\|u_r\|_1 = \int_{\mathbb{T}} u(r\zeta) d\lambda(\zeta) = u(0).$$

Hence, it follows from (i) that $u = \mathscr{P}[\mu]$ for some complex Borel measure μ and μ is the w^* limit of $\mu_{r_n} = u_{r_n} d\lambda$. Since μ_{r_n} are non-negative measures, it follows that μ is non-negative, since the property to assign non-negative values to non-negative continuous functions is preserved when taking w^*-limits.

(iii) $\Longrightarrow$: Let $f \in L^\infty(\mathbb{T})$. Since $P(z,\zeta)d\lambda$ is a probability measure, we obtain

$$|u_r(z)| \leq \int_{\mathbb{T}} |f(\zeta)| P(rz,\zeta) d\lambda(\zeta) \leq \|f\|_\infty.$$

(iv) $\Longrightarrow$: Assume now that $f \in C(\mathbb{T})$. Then

$$\tilde{u} := \begin{cases} \mathscr{P}[f](z), & z \in \mathbb{D}, \\ f(z), & z \in \mathbb{T} \end{cases}$$

is continuous on $\overline{\mathbb{D}}$ by Theorem 3.14 and hence uniformly continuous. This implies that $\lim\limits_{r\nearrow 1} \|u_r - f\|_\infty = 0$.

(ii) $\Longrightarrow$: By Jensens inequality for the measure $P(rz,\zeta)d\lambda(z)$ and the convex function $|\cdot|^p$, for $z \in \mathbb{T}$ we have

$$|u_r(\zeta)|^p = \left| \int_{\mathbb{T}} f(\zeta) P(rz,\zeta) d\lambda(\zeta) \right|^p \leq \int_{\mathbb{T}} |f(\zeta)|^p \, P(rz,\zeta) d\lambda(\zeta) = \mathscr{P}[|f|^p](rz).$$

Since $\mathscr{P}[|f|^p](rz)$ is harmonic, it follows from the mean value property that

$$\|u_r\|_p^p \leq \int_{\mathbb{T}} \mathscr{P}[|f|^p](rz) d\lambda(z) = \mathscr{P}[|f|^p](0) = \int_{\mathbb{T}} |f(\zeta)|^p \, d\lambda(\zeta) = \|f\|_p^p. \tag{3.6}$$

It remains to show that $\|u_r - f\|_p \to 0$. Fix $\epsilon > 0$ and choose $g \in C(\mathbb{T})$ with $\|f - g\|_p < \epsilon$ and set $v = \mathscr{P}[g]$. Then we know from ((iv) $\Longrightarrow$) that $\lim\limits_{r\to 1} \|v_r - g\|_\infty = 0$. Hence

$$\limsup_{r\to 1} \|v_r - g\|_p \leq \lim_{r\to 1} \|v_r - g\|_\infty = 0.$$

Therefore, we find r_0 such that for $r_0 < r < 1$ $\|v_r - g\|_p < \epsilon$. As in (3.6) (for the function $f - g$) we see that

$$\|u_r - v_r\|_p \leq \|f - g\|_p < \epsilon.$$

Thus it follows that for $r_0 < r < 1$

$$\|u_r - f\|_p \leq \|u_r - v_r\|_p + \|v_r - g\|_p + \|g - f\|_p < 3\epsilon.$$

Since ϵ was arbitrary, the claim follows.

(iv) $\Longleftarrow$: If $\lim_{r\nearrow 1} \|u_r - f\|_\infty = 0$ for some continuous f, then for $p = 2$ we have

$$\|u_r\|_{L^2(\mathbb{T})} \leq \|u_r - f\|_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})} \leq \|u_r - f\|_\infty + \|f\|_{L^2(\mathbb{T})}.$$

Thus, $\sup_{r\in[0,1)} \|u_r\|_{L^2(\mathbb{T})} < \infty$. Hence $u = \mathscr{P}[g]$ for some $g \in L^2(\mathbb{T})$ and $\|u_r - g\|_{L^2(\mathbb{T})} \to 0$. On the other, since

$$\|u_r - f\|_{L^2(\mathbb{T})} \leq \|u_r - f\|_\infty,$$

it also follows that $\|u_r - f\|_{L^2(\mathbb{T})} \to 0$ and hence $f = g$.

Uniqueness: It remains to show uniqueness: Let μ be a complex measure and set $v = \mathscr{P}[\mu]$. We need to show that $v = 0$ implies $\mu = 0$. Choose $f \in C(\mathbb{T})$ and set $u = \mathscr{P}[f]$. Since for $\zeta, z \in \mathbb{T}$, $P(r\zeta, z) = P(rz, \zeta)$ it follows from Fubini that

$$\int_{\mathbb{T}} u_r(\zeta) d\mu(\zeta) = \int_{\mathbb{T}} v_r(z) f(z) f\lambda(z).$$

Since $v = 0$ also $v_r = 0$ and since $\lim_{r\nearrow} \|u_r - f\|_\infty = 0$ we conclude that

$$\int_{\mathbb{T}} f d\mu = 0, \tag{3.7}$$

for every $f \in C(\mathbb{T})$. Recall that by the Riesz-Representation theorem $C(\mathbb{T})^*$ is the set of all complex Borel measures on $\mathbb{T}$. If we denote the functional defined by (3.7) with Λ_μ, then since Riesz Representation gives an isometry, we know that $\|\Lambda_\mu\| = |\mu|(\Omega)$. Thus, we conlude from (3.7) that $\mu = 0$. $\square$

Let us point out that finding f in the proof of (ii) fails if $p = 1$, since $L^1(\mathbb{T})$ is not reflexive.

If u is harmonic in the dis and for $p > 1$ we have $\sup_{r\in[0,1)} \|u_r\|_p < \infty$, then there exists $f \in L^p$ such that $u = \mathscr{P}[f]$. The theorem of Fatou now says that u has radial limits u^* a.e. and $u^* = f$ a.e.. That is, u can be represented as the Poisson integral of its boundary values. This is no longer true, of $p = 1$ as the following example shows:

Example 3.28. Let $\mu = \delta_{\{1\}}$ and $u = \mathscr{P}[\mu]$. Then from theorem 3.27i we conclude that $\sup_{r\in[0,1)} \|u_r\|_1 < \infty$ and by the theorem of Fatou we get that for every $z \neq 1$

$$\lim_{r\to 1} \mathscr{P}[u](rz) = 0.$$

That is, u is not the Poisson integral of its a.e. boundary values. ◀

We get the following two immediate representation theorems for analytic functions:

Corollary 3.29. *Let $f \in \mathrm{Hol}(\mathbb{D})$ and assume that*

$$\sup_{r\in[0,1)} \int_{\mathbb{T}} |\operatorname{Re} f_r| d\lambda < \infty.$$

Then there exists μ and $c \in \mathbb{R}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + ic, \quad z \in \mathbb{D}. \tag{3.8}$$

It holds that $c = \operatorname{Im} f(i)$.

Proof. By Theorem 3.27, we find a real measure μ so that

$$\operatorname{Re} f = \mathscr{P}[\mu].$$

Since

$$\operatorname{Re}\left(f - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right) = \operatorname{Re} f - \mathscr{P}[\mu] = 0$$

we find $c \in \mathbb{R}$ such that (3.8) holds. □

Similarly we get the following:

Corollary 3.30 (Riesz-Herglotz). *Let $f \in \mathrm{Hol}(\mathbb{D})$ and $\operatorname{Re} f \geq 0$. Then there exists a unique measure μ and $c \in \mathbb{R}$ such that*

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + ic, \quad z \in \mathbb{D}.$$

It holds that $c = \operatorname{Im} f(i)$.

Combining (i) and (v) we get

Corollary 3.31. *Let u be harmonic in $\mathbb{D}$ and $u \geq 0$. Then*

$$\sup_{r\in[0,1)} \int_{\mathbb{T}} u_r(\zeta) d\lambda(\zeta) < \infty.$$

3.5.1. Subharmonic function. Before we introduce subharmonic functions, let us recall/prove some facts about semicontinuous functions.

Definition 3.32. *Let $\Omega \subset \mathbb{C}$. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called upper semicontinuous, if for every $m \in \mathbb{R}$ the set $u^{-1}([m, \infty))$ is closed in Ω.*

We collect some properties about upper semicontinuous functions:

Lemma 3.33.
(i) u is upper semicontinuous if and only if for every $w \in \Omega$ $\limsup_{z\to w} u(z) \leq u(w)$;
(ii) If $\lambda, \mu \geq 0$ and u, v upper semicontinuous, then so is $\lambda u + \mu v$;
(iii) If $(u_i)_{i\in I}$ is a family of upper semicontinuous functions, then $\inf_{i\in I} u_i$ is upper semicontinuous. If I is finite, then also $\max_{i\in I} u_i$ is upper semicontinuous;

Proof. (i): Assume that there exists a sequence z_n, $z_n \to w$ and $M := \lim_{n\to\infty} u(z_n) > u(w)$. Choose $m \in (u(w), M)$. Then for n sufficiently large, we have $z_n \in u^{-1}([m, \infty))$. Since $u^{-1}([m, \infty))$ is closed, it follows that $w \in u^{-1}([m, \infty))$. A contradiction.

On the other hand, choose $m \in \mathbb{R}$ and $z_n \in u^{-1}([m,\infty))$ and $z_n \to w$. Then,

$$u(w) \geq \limsup_{n\to\infty} u(z_n) \geq m$$

and hence $w \in u^{-1}([m,\infty))$.

(ii): This is obvious from (i).

(iii): Define $u := \inf_{i\in I} u_i$. Then it follows that $u^{-1}([m,\infty)) = \cap_{i\in I} u_i([m,\infty))$. Moreover, if $v = \max_{1\leq i\leq n} u_i$, then $v^{-1}([m,\infty)) = \cup_{i=1}^{n} u_i([m,\infty))$. □

LEMMA 3.34. *Assume that u is upper semicontinuous and $K \subset \Omega$ compact. Then u is bounded on K and attains its maximum.*

PROOF. Let K be compact subset of Ω. Since $u^{-1}([n,\infty))$ is closed, for all n the set $V_n := \{z \mid u(z) < n\}$ is open and $(V_n)_{n\in\mathbb{N}}$ form an open cover of K. Thus, it has a finite subcover, and we conclude that u is bounded. Let $M = \sup_{z\in K} u(z)$. Then $U_n := \{z \mid u(z) < M - 1/n\}$ cannot cover K, since they have no finite subcover. Thus, there must be $z \in K$ with $u(z) \geq M - 1/n$ for all n. Thus, $u(z) \geq M$ and hence by definition of M $u(z) = M$. □

The following lemma will be useful later.

LEMMA 3.35. *Assume that $u : \Omega \to [-\infty,\infty)$ is upper semicontinuous and $\sup_{z\in K} u(z) = M$. Then there exist $\phi_n \in C(\Omega,\mathbb{R})$ such that for all $x \in \Omega$ $M \geq \phi_1(x) \geq \phi_2(x) \geq \cdots$. Moreover, for $x \in K$ and $n \in \mathbb{N}$ we have $\phi_n(x) \geq u(x)$ and $\lim_{n\to\infty} \phi_n(x) = u(x)$.*

PROOF. We can suppose $u \not\equiv -\infty$. Otherwise take $\phi_n = -n$. Define $\phi_n : \Omega \to \mathbb{R}$ by

$$\phi_n(x) = \sup_{y\in K}(u(y) - n|x-y|)$$

Clearly we have $M \geq \phi_n(x)$ on Ω and $\phi_n(x) \geq u(x)$ on K. Then we have

$$|\phi_n(x) - \phi_n(x')| \leq n|x - x'|$$

and thus, ϕ_n is continuous. In particular $\lim\limits_{n\to\infty} \phi_n = \phi(x)$ exists and $\phi \geq u$. Assume that there is $a \in K$ with $\phi(a) > u(a)$. Choose $m \in (u(a), \phi(a))$. Then, by the definition of ϕ_n, there exists $y_n \in K$ with $u(y_n) - n|a - y_n| \geq m$, i.e.,

$$m + n|a - y_n| \leq u(y_n) \leq M.$$

In particular, $n|a-y_n|$ is bounded and thus $y_n \to a$. It now follows from upper semicontinuity that

$$m \leq \limsup_{n\to\infty} u(y_n) \leq u(a).$$

A contradiction. □

3.5.2. Subharmonic functions. In spirit, at least, a function u is subharmonic if its Laplacian satisfies $\Delta u \geq 0$. But we do not require this level of smoothness here. Instead we proceed by analogy with convex functions on $\mathbb{R}$. If $\psi \in C^2(\mathbb{R})$, then ψ is convex iff $\psi'' \geq 0$. However, convexity is defined via a submean inequality that allows also $\psi(t) = |t|$ to be convex. There is however a technical issue. Convex functions on open intervals are automatically continuous. For subharmonic function we have to add certain regularity. It turns out, that the right degree of regularity is upper semicontinuity.

Definition 3.36. *Let $\Omega \subset \mathbb{C}$ be open. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic, if*

(i) u is upper semicontinuous;

(ii) u satisfies the local submean inequality, i.e., for every $w \in \Omega$ there exists $R > 0$ such that for $0 \le r < R$ it holds that

$$u(w) \le \int_{-\pi}^{\pi} u(w + e^{it}) \frac{dt}{2\pi}.$$

A function u is superharmonic, if $-u$ is subharmonic.

It follows directly from this definition that u is harmonic, iff u is sub- and superharmonic.

Note that the integral is well defined (although it may be $-\infty$), since u is bounded from above on compacts. Note also, that subharmonicity is again defined as a local property of u.

We collect some properties of subharmonic functions below.

Proposition 3.37. *Let Ω be open and u, v subharmonic. Then*

(i) For $\alpha, \beta \ge 0$, also $\alpha u + \beta v$ are subharmonic;

(ii) $\max\{u, v\}$ are subharmonic;

(iii) If u attains a (global) maximum on Ω, then u is constant;

(iv) If Ω is bounded and for every $\zeta \in \partial\Omega$ and assume that u is upper semicontinous on $\overline{\Omega}$ and $u|_{\partial\Omega} \le 0$, then $u \le 0$ in Ω.

Proof. All proofs are either straight forward or can be carried out similarly to previous proofs. □

Note that we indeed require a global maximum here. Also a subharmonic function can have a global minimum without being constant. As an example take $u(z) = \max\{\operatorname{Re} z, 0\}$ on $\mathbb{C}$ which has a local maximum and a global minimum on $\mathbb{C}$ without being constant.

Example 3.38. If $\Omega \subset \mathbb{C}$ is open and $f \in \operatorname{Hol}(\Omega)$, then $\log|f|$ and $\log^+|f|$ are subharmonic, where for $a \ge 0$, $\log^+ a = \max\{a, \log a\}$.

Clearly, $u = \log|f|$ is upper semicontinuous. We have already remarked that subharmonicity is a local property. If $f(w) \neq 0$, then there is $r > 0$, so that f does not vanish on $B_r(w)$. But then $\log f$ is analytic and $\operatorname{Re}\log f = \log|f|$ is even harmonic. If $f(w) = 0$, then $u(w) = -\infty$, and the submean inequality is clear. Since $\log^+|f| = \max\{\log|f|, 0\}$, also $\log^+|f|$ is subharmonic. ◀

We have the following characterization of subharmonic functions, which also explains the name:

Theorem 3.39. *Let $\Omega \subset \mathbb{C}$ be open and $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. Then the following are equivalent:*

(i) u is subharmonic on Ω;

(ii) For every w, R such that $\overline{B_R(w)} \subset \Omega$ it holds that for every $r < R$ and $\theta \in \mathbb{R}$

$$u(w + re^{i\theta}) \le \int_{-\pi}^{\pi} P_{R,w}(w + re^{i\theta}, w + Re^{it}) u(w + Re^{it}) \frac{dt}{2\pi}.$$

(iii) For every $D \subset \Omega$ open and $\overline{D} \subset \Omega$ compact and h harmonic in D and continuous on $\overline{D}$ such that $u \le h$ on ∂D it follows that $u \le h$ in D.

PROOF. (i) $\implies$ (iii): The function $u - h$ is subharmonic. So the claim follows by the maximum principle for subharmonic functions.

(iii) $\implies$ (ii): Fix R, w such that $\overline{B_R(w)} \subset \Omega$. We find continuous $\phi_n : \partial B_R(w) \to \mathbb{R}$ such that $\phi_n \searrow u$ pointwise. Moreover,

$$\Phi_n(z) = \begin{cases} \int_{\partial B_R(w)} P_{R,w}(z,\zeta)\phi_n(\zeta)d\lambda(\zeta), & |z-w| < R \\ \phi_n(z), & |z-w| = R \end{cases}$$

are continuous on $\overline{B_R(w)}$ and harmonic on $B_R(w)$. Hence, on $\overline{B_R(w)}$

$$u(z) \leq \Phi_n(z).$$

Since $P_{R,w}(z,\zeta)\phi_n(\zeta) \searrow P_{R,w}(z,\zeta)u(\zeta)$, the monotone convergence theorem implies that

$$\int_{\partial B_R(w)} P_{R,w}(z,\zeta)\phi_n(\zeta)d\lambda(\zeta) \to \int_{\partial B_R(w)} P_{R,w}(z,\zeta)u(\zeta)d\lambda(\zeta).$$

(ii) $\implies$ (i): Set $r = 0$, since $P_{R,w}(w,\zeta) = 1$. □

As a corollary, we obtain from (ii) that u also satisfies a global submean inequality.

Let dA denote the two-dimensional Lebesgue measure on $\mathbb{C}$. Since upper semicontinuous functions are bounded above on compact subsets of Ω, clearly

$$\int_K u dA < \infty.$$

In fact, due to the submean inequality, we see that if $u \not\equiv -\infty$, also $\int_K u dA > -\infty$.

THEOREM 3.40. *Let $\Omega \subset \mathbb{C}$ be a domain and u be subharmonic on Ω, with $u \not\equiv -\infty$. Then, for every $K \subset \Omega$ compact, we have*

$$\int_K |u| dA < \infty.$$

PROOF. By compactness, it suffices to show that for any $w \in \Omega$, there exists $r > 0$ such that

$$\int_{B_r(w)} |u| dA < \infty. \tag{3.9}$$

Let us define

$$A = \{w \mid \exists r > 0 \text{ such that (3.9) holds}\}, \quad B = \{w \mid \forall r > 0 \text{ (3.9) does not hold}\}.$$

Again, we show that A, B are open and $u|_B = -\infty$. From this the result follows from connectedness.

That A is open follows from the definition. Since $B_r(w)$ is open and (3.9) is increasing w.r.t. set inclusion.

B is open: Let $w \in B$ and choose $r > 0$ such that $\overline{B_{3r}(w)} \subset \Omega$. Then we have

$$\int_{B_r(w)} u dA = -\infty.$$

Chooce $w' \in B_r(w)$ and $r' = r + |w - w'|$. Then $B_r(w) \subset B_{r'}(w') \subset \overline{B_{3r}(w)} \subset \Omega$. Therefore,

$$\int_{B_{r'}(w')} u dA = -\infty$$

By the submean inequality, for every, $0 \leq \rho \leq r'$ we have

$$u(w') \leq \int_{-\pi}^{\pi} u(w' + \rho e^{it}) \frac{dt}{2\pi}$$

Thus, multiplying by $2\pi\rho$ and integrating ρ from 0 to r' we get

$$\pi {r'}^2 u(w') = 2\pi \int_0^r u(w') d\rho \leq \int_0^{r'} \int_{-\pi}^{\pi} u(w' + \rho e^{it}) dt d\rho = \int_{B_{r'}(w')} u dA = -\infty.$$

Thus, $u(w') = -\infty$ in $B_r(w)$. This finishes the proof. $\square$

Corollary 3.41. *Let u be subharmonic on a domain $\Omega \subset \mathbb{C}$ and $u \not\equiv -\infty$. If $\overline{B_R(w)} \subset \Omega$, then*

$$\int_{-\pi}^{\pi} u(w + Re^{it}) \frac{dt}{2\pi} > -\infty.$$

Proof. Fix R, w so that $\overline{B_R(w)} \subset \Omega$. Since u is bounded there, by subtracting a constant we can assume that $u|_{\overline{B_R(w)}} \leq 0$. Recall that for $r < R$ we have

$$P_{R,w}(w + re^{i\theta}, w + Re^{it}) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t - \theta)}$$

and

$$\frac{R - r}{R + r} \leq \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t - \theta)}.$$

Thus,

$$u(w + re^{i\theta}) \leq \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t - \theta)} u(w + Re^{it}) \frac{dt}{2\pi} \tag{3.10}$$

$$\leq \frac{R - r}{R + r} \int_{-\pi}^{\pi} u(w + Re^{it}) \frac{dt}{2\pi}. \tag{3.11}$$

Hence, if the last integral were $-\infty$, we would conlude that $u|_{B_R(w)} \equiv -\infty$. A contradiction. $\square$

We require 3 Lemmata about families of subharmonic functions:

Lemma 3.42. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of subharmonic functions on an open set Ω such that $u_1 \geq u_2 \geq \dots$. Then $u := \lim_{n\to\infty} u_n$ is subharmonic.*

Proof. Since $u = \inf_{n\in\mathbb{N}} u_n$, it follows from Lemma 3.33 that u is upper semincontinuous. The submean inequality follows again from the monotone convergence theorem. $\square$

Lemma 3.43. *Let X be a compact topological space, Ω open subset of $\mathbb{C}$ and $v : \Omega \times \mathbb{C} \to [-\infty, \infty)$ such that*

(i) v is upper semicontinuous on $\Omega \times X$;
(ii) For every $x \in X$, the map $z \mapsto v(z, x)$ is subharmonic;

Then $u(z) = \sup_{x\in X} v(z, x)$ is subharmonic.

Proof. We start with showing that v is upper semincontinuous, by showing that for every $m \in \mathbb{R}$ $L_m := \{z \mid u(z) < m\}$ is open. Choose $z \in L_m$, i.e., $u(z) < m$. By definition, $v(x) < m$ for all $x \in X$. Since v is upper semicontinuous, there exists r_x and vicinities N_x such that $B(z, r_x) \times N_x \subset L_m$. Since X is compact, there exists $x_1, \dots, x_n$ such that $(N_{x_k})_{k=1}^n$ cover X. Define $r' = \min\{r_{x_1}, \dots, r_{x_n}\} > 0$. Fix $w \in B_{r'}(w)$. Then

$$u(w) = \sup_{x \in x} u(w, x) = \sup_{\substack{x \in N_{x_k}, \\ 1 \le k \le n}} u(w, x) < m.$$

and hence N_m is open.

Now suppose that $\overline{B_r(z)} \subset \Omega$. Then for each $x \in X$ we have

$$v(z, x) \le \int_{-\pi}^{\pi} v(z + re^{it}, x) \frac{dt}{2\pi} \le \int_{-\pi}^{\pi} u(z + re^{it}) \frac{dt}{2\pi}$$

Taking the supremum of $x \in X$ we get that u satisfies the submean inequality. □

Lemma 3.44. *Let (Y, μ) be a measure space with $\mu(Y) < \infty$, $\Omega \subset \mathbb{C}$ open, and $v : Y \times \Omega \to [-\infty, \infty)$ be such that*

(i) v is measureable on $Y \times \Omega$;
(ii) For every $y \in Y$, the map $z \mapsto v(z, y)$ is subharmonic;
(iii) The map $z \mapsto \sup_{y \in Y} v(z, y)$ is locally bounded in Ω.

Then $u(z) = \int_Y v(z, y) d\mu(y)$ is subharmonic on Ω.

Proof. As we have mentioned that subharmonicity is a local property, it is sufficient to show that u is subharmonic on $B_r(w)$ with $\overline{B_r(w)} \subset \Omega$. Then (iii) implies that $\sup_{y \in Y} v(z, y)$ is bounded on $B_r(w)$ and by subtracting a constant we can assume that it is $v \le 0$ on $B_r(w) \times Y$. This justifies the application of Fubini's and Fatou's theorem in what follows. Let $B_w(z) \ni z_n \to z$

$$\begin{aligned} \limsup_{n \to \infty} u(z_n) &= \limsup_{n \to \infty} \int_Y v(z_n, y) d\mu(y) \\ &\le \int_Y \limsup_{n \to \infty} v(z_n, y) d\mu(y) \le \int_Y \limsup_{n \to \infty} v(z, y) d\mu(y) = u(z) \end{aligned}$$

and thus u is upper semincontinuous. Moreover, by Fubini's theorem

$$\begin{aligned} \int_{-\pi}^{\pi} u(z + re^{it}) \frac{dt}{2\pi} &= \int_{-\pi}^{\pi} \int_Y v(z + re^{it}, y) d\mu(y) \frac{dt}{2\pi} \\ &= \int_Y \int_{-\pi}^{\pi} v(z + re^{it}, y) \frac{dt}{2\pi} d\mu(y) \ge \int_Y v(z, y) d\mu(y) = u(z). \end{aligned}$$

□

The connection to convex functions mentioned in the beginning was no coincidence. In fact there is a close connection between subharmonic and convex functions.

Definition 3.45. *Let $-\infty \le a < b \le \infty$. A function $\psi : (a, b) \to \mathbb{R}$ is called convex, if for every $x, y \in (a, b)$ and $0 \le t \le 1$ it holds that*

$$\psi(tx + (1 - t)y) \le t\psi(x) + (1 - t)\psi(y).$$

The following lemma allows us to construct many subharmonic functions:

LEMMA 3.46. *Let $\Omega \subset \mathbb{C}$ be open, $-\infty \le a < b \le \infty$ and $u : \Omega \to [a,b)$ subharmonic and $\psi : (a,b) \to \mathbb{R}$ be an increasing and convex function. Then $\psi \circ u$ is subharmonic, where $\psi(a) = \lim_{t\to a} \psi(t)$.*

PROOF. Let a_n in (a,b) with $a_n \searrow a$ and set $u_n = \max\{u, a_n\}$. Then, since ψ is continuous on (a,b), $\psi \circ u_n$ is upper semicontinuous. Also, if $\overline{B_r(w)} \subset \Omega$, then by Jensen's inequality

$$\psi \circ u_n(w) \le \psi\left(\int_{-\pi}^{\pi} u_n(w + re^{it})\frac{dt}{2\pi}\right) \le \int_{-\pi}^{\pi} (\psi \circ u_n)(w + re^{it})\frac{dt}{2\pi}.$$

That is, $\psi \circ u_n$ is subharmonic. Since $\psi \circ u_n \searrow \psi \circ u$, it follows from Lemma 3.42 that $\psi \circ u$ is subharmonic. □

COROLLARY 3.47. *If u is subharmonic on $\Omega \subset \mathbb{C}$ open, then also $\exp u$.*

EXAMPLE 3.48. If f is analytic in Ω, then for $p > 0$, $|f|^p$ is subharmonic. We have $|f|^p = \exp(p \log |f|)$. ◀

We say that a function $f : (0,R) \to \mathbb{R}$ is a convex function of $\log r$, if for all $r_1, r_2 \in (0,R)$, $0 \le \lambda \le 1$, we have

$$f(r) \le (1-\lambda) f(r_1) + \lambda f(r_2), \quad \log r = (1-\lambda)\log r_1 + \lambda \log r_2.$$

That is, the function $g(t) = f(e^t)$ is convex on the corresponding domain.

LEMMA 3.49. *Let $v : \mathbb{D} \to [-\infty, \infty)$ be radial, (i.e., $v(z) = v(|z|)$) and assume that $v \not\equiv -\infty$. Then v is subharmonic on $\mathbb{D}$ if and only if*

(i) $v(r)$ is increasing;

(ii) $v(r)$ is a convex function of $\log r$;

(iii) $\lim_{r\to 0} v(r) = v(0)$.

PROOF. $\Longleftarrow$: This follows from Lemma 3.46 with the subharmonic function $u(z) = \log |z|$ and the increasing convex function $\psi(t) = v(e^t)$.

$\Longrightarrow$: Assume that v is subharmonic on $\mathbb{D}$. The maximum principle Proposition 3.37 we get

$$v(r_1) \le \sup_{|z|=r_2} v(z) = v(r_2)$$

and thus v is increasing. In the same way we see that $v(0) \le v(r)$ and thus, in particular

$$v(0) \le \liminf_{r\to 0} v(r).$$

On the other hand, upper semicontinuity imiplies

$$v(0) \ge \limsup_{r\to 0} v(r)$$

and thus $\lim_{r\to 0} v(r) = v(0)$.

It remains to show that v is a convex function of $\log r$. Notice that since v is radial, by Corollary 3.41 we have $v(r) > -\infty$ for every $r > 0$. Choose $0 < r_1 < r_2 < 1$ and $\alpha, \beta \in \mathbb{R}$ so that

$$h(r) := \alpha + \beta \log r$$

satisfies $h(r_1) = v(r_1)$ and $h(r_2) = v(r_2)$. Note that on the annulus $A_{r_1,r_2} := \{z \mid r_1 < |z| < r_2\}$, $\alpha + \beta \log |z|$ is harmonic and we can thus apply the maximum principle to $v(z) - h(|z|)$. Since $v(z) - h(|z|) \leq 0$ on $\partial A_{r_1,r_2}$ we conclude that

$$v(r) \leq \alpha + \beta \log r, \quad r_1 \leq r \leq r_2.$$

Hence, if $0 \leq \lambda \leq 1$ and $\log r = (1-\lambda)\log r_1 + \lambda \log r_2$, then

$$\begin{aligned} v(r) \leq & \alpha + \beta \log r \\ =&(1-\lambda)(\alpha + \beta \log r_1) + \lambda(\alpha + \beta \log r_2) \\ =&(1-\lambda)v(r_1) + \lambda v(r_2). \end{aligned}$$

□

Let u be a subharmonic function on $\mathbb{D}$ with $u \not\equiv -\infty$. Then let us define for $0 < r < 1$

$$\begin{aligned} T(r) &= \sup_{|z|=r} u(z), \\ M(r) &= \int_{-\pi}^{\pi} u(re^{it}) \frac{dt}{2\pi}, \\ F(r) &= \int_{B_r(0)} u \frac{dA}{\pi r^2}. \end{aligned}$$

Note that all this quantities are finite. Moreover,

$$F(r) = \frac{2}{r^2} \int_0^r M(s) s ds. \tag{3.12}$$

Let us start with a preliminary observation. Define

$$\tilde{u}(z,t) = u(ze^{it}), \quad u_3(z,s,t) = u(zse^{it})$$

and

$$\begin{aligned} v_1(z) &= \sup_{t \in [-\pi,\pi]} \tilde{u}(z,t), \\ v_2(z) &= \int_{-\pi}^{\pi} \tilde{u}(z,t) \frac{dt}{2\pi}, \\ v_3(z) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \int_0^1 u_3(z,s,t) s ds dt. \end{aligned}$$

Then by Lemma 3.43 and 3.44 v_1, v_2, v_3 are subharmonic and by definition radial and

$$v_1(r) = T(r), \quad v_2(r) = M(r), \quad v_3(r) = F(r).$$

Thus, we can apply 3.49 to obtain:

Theorem 3.50. *We have*

(i) T, M, F *are increasing;*
(ii) T, M, F *are convex functions of* $\log r$*;*
(iii) $T(r) \geq M(r) \geq F(r) \geq u(0)$*;*
(iv) $\lim_{r\to 0} T(r) = \lim_{r \to 0} M(r) = \lim_{r\to 0} F(r) = u(0)\cdot$

PROOF. (i),(ii) follow immediately by the discussion above.

(iii): The first inequality is clear. Now, by (i) and the submean inequality, we have for $0 < s < r < 1$

$$u(0) \leq M(s) \leq M(r)$$

Multiplying by $2s/r^2$ and integrating s from 0 to r we get

$$u(0) \leq \frac{2}{r^2} \int_0^r M(s) s ds \leq M(r).$$

Hence, by (3.12) we have $u(0) \leq F(r) \leq M(r)$. This shows (iii).

(iv): It suffices to show that $\limsup_{r \to 0} T(r) \leq u(0)$. But this follows immediately from upper semicontinuity of u. $\square$

CHAPTER 4

H^p spaces

4.1. Functions of bounded characteristic

Before introducing H^p spaces we will discuss a more general class. So called functions of bounded characteristics.

We will now focus on functions analytic in $\mathbb{D}$.

DEFINITION 4.1. *Let $f \in \mathrm{Hol}(\mathbb{D})$ and $f_r(\zeta) = f(r\zeta)$, $\zeta \in \mathbb{D}$. We write $f \in N = N(\mathbb{D})$ (and say f is of bounded characteristic), if*

$$\|f\|_0 := \sup_{0\le r<1} \int_{\mathbb{T}} \log^+ |f_r(\zeta)| d\lambda(\zeta) < \infty.$$

Note that $\log^+|f|$ is subharmonic and therefore

$$\int_{\mathbb{T}} \log^+ |f_r(\zeta)| d\lambda(\zeta)$$

is increasing in r. Thus, the $\sup_{0\le r<1}$ could be replaced by $\lim_{r\nearrow 1}$. Despite the notation, $\|f\|_0$ does not define a norm. However, we will see that N forms a vector space. The following theorem is a fundamental characterization of the class N. It is also the reason, why the class N is also sometimes called functions of bounded type.

THEOREM 4.2 (R. Nevanlinna). *Let $f \in \mathrm{Hol}(\mathbb{D})$. Then $f \in N$ if and only if $f = \varphi/\psi$, where φ and ψ are bounded analytic functions in $\mathbb{D}$. In this case, one can choose φ, ψ so that ψ does not vanish on $\mathbb{D}$ and $|\psi| \le 1$ and $|\varphi| \le 1$ on $\mathbb{D}$.*

PROOF. Let $f = \varphi/\psi$ and without loss of generality we can assume that $|\varphi| \le 1, |\psi| \le 1$ on $\mathbb{D}$ and ψ has no zeros. We have for any $r < 1$ and $\zeta \in \mathbb{D}$

$$\log^+ \left| \frac{\varphi(r\zeta)}{\psi(r\zeta)} \right| \le -\log|\psi(r\zeta)|.$$

Since ψ has no zeros, $\log|\psi(r\zeta)|$ is harmonic in a vicinity of $\mathbb{D}$. Thus we obtain

$$\int_{\mathbb{T}} \log^+ \left| \frac{\varphi(r\zeta)}{\psi(r\zeta)} \right| d\lambda(\zeta) \le -\int_{\mathbb{T}} \log|\psi(r\zeta)| d\lambda(\zeta) = -\log|\psi(0)|.$$

Conversely, assume that $f \in N$. Consider the measure $\log^+|f_r(\zeta)|d\lambda(\zeta)$ as elements of $C(\mathbb{T})^*$. Then we have

$$\|\log^+|f_r(\zeta)|d\lambda(\zeta)\| = \int_{\mathbb{T}} \log^+|f_r(\zeta)|d\lambda(\zeta) \le \|f\|_0$$

and thus, they are contained in a ball of radius $\|f\|_0$. By the Banach-Alaoglu theorem, there exists $r_n \to 1$ and a positive finite (in fact complex, but since all measure are positive, this also holds for the weak-$*$ limit) measure μ such that in the weak-$*$ sense

$$\lim_{n\to\infty} \log^+ |f_r(\zeta)| d\lambda(\zeta) = \mu$$

Define

$$\psi(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta - z}{\zeta + z} d\mu(\zeta) \right), \quad z \in \mathbb{D}.$$

Then, ψ is analytic has no zeros and is bounded by 1 in $\mathbb{D}$, since

$$\log |\psi(z)| = -\mathscr{P}[\mu](z) \leq 0.$$

We want to show

$$|f(z)| \leq \frac{1}{|\psi(z)|}. \tag{4.1}$$

Fix $r < 1$ and consider the function $u(z) = \log |f(rz)|$. Note that u is subharmonic in a vicinity of $\mathbb{D}$ and that $u(\zeta) \leq \log^+ |f(r\zeta)|$ on $\mathbb{T}$. Thus, by Theorem 3.39

$$\log |f(rz)| = u(z) \leq \mathscr{P}[\log^+ |f(r\zeta)|](z), \quad z \in \mathbb{D}.$$

Fix $z \in \mathbb{D}$ in the above inequality. Sending $r_n \to 1$ we obtain

$$\log |f(z)| \leq \mathscr{P}[\mu](z)$$

and thus (4.1) follows. Setting $\varphi = f\psi$ proves the desired representation for f. □

Corollary 4.3. *Let $f \in N$, $f \not\equiv 0$, then*

$$\limsup_{r \nearrow 1} \int_{\mathbb{T}} |\log |f_r(\zeta)|| d\zeta < \infty$$

Proof. Let first $g \in \mathrm{Hol}(\mathbb{D})$ with $|g| \leq 1$ on $\mathbb{D}$, $g \not\equiv 0$. Then $|\log |g_r(\zeta)|| = -\log |g_r(\zeta)|$. Moreover, since $\log |g(\zeta)|$ is subharmonic, $\int_{\mathbb{T}} \log |g_r(\zeta)| d\zeta$ is monotonic increasing and bounded above by 0. Hence,

$$\lim_{r \nearrow 1} \int_{\mathbb{T}} \log |g_r(\zeta)| d\zeta = -C \leq 0.$$

Combining this, we get

$$\lim_{r \nearrow 1} \int_{\mathbb{T}} |\log |g_r(\zeta)|| d\zeta = C \in [0, \infty).$$

Now write $f = \varphi/\psi$ as in Theorem 4.2. Then we have

$$\limsup_{r \nearrow 1} \int_{\mathbb{T}} |\log |f_r(\zeta)|| d\zeta \leq \limsup_{r \nearrow 1} \int_{\mathbb{T}} |\log |\varphi_r(\zeta)|| d\zeta + \limsup_{r \nearrow 1} \int_{\mathbb{T}} |\log |\psi_r(\zeta)|| d\zeta < \infty.$$

□

As an immediate corollary we obtain that N is a vector space:

Corollary 4.4. *For every $\alpha \in \mathbb{C}$, we have that $\alpha \in N$. Let $f, g \in N$, then $f + g, f \cdot g \in N$. If $\frac{f}{g} \in \mathrm{Hol}(\mathbb{D})$, then we have $\frac{f}{g} \in N$.*

Theorem 4.5. *Let $f \in N$. Then for Lebesgue a.e. $\zeta \in \mathbb{T}$, the nontangential boundary values*

$$f(\zeta) = \lim_{z \hat{\to} \zeta} f(z).$$

If $f \not\equiv 0$, then $\log |f^*| \in L^1(\mathbb{T})$.

Proof. Let $g \in \mathrm{Hol}(\mathbb{D})$, $g \not\equiv 0$ and $|g| \le 1$. Then, by Theorem 3.27, Corollary 3.23 and Proposition 3.26, g has nontangential boundary values a.e. on $\mathbb{T}$. Since $\log |g|$ is subharmonic in $\mathbb{D}$, $\int_{\mathbb{T}} \log |g_r(\zeta)| d\zeta$ is increasing in r and bounded above by 0. In particular

$$\lim_{r \nearrow 1} \int_{\mathbb{T}} \log |g_r(z)| = C \le 0.$$

The Lemma of Fatou implies that

$$\begin{aligned} -\int_{\mathbb{T}} \log |g^*(\zeta)| d\lambda(\zeta) &= \int_{\mathbb{T}} \liminf_{r \nearrow 1} -\log |g_r(\zeta)| d\lambda(\zeta) \le \liminf_{r \nearrow 1} -\int_{\mathbb{T}} \log |g_r(z)| d\lambda(\zeta) \\ &= -\lim_{r \nearrow 1} \int_{\mathbb{T}} \log |g_r(z)| d\lambda(\zeta) = -C. \end{aligned}$$

Note that this in particular implies that $g^* \neq 0$ a.e. on $\mathbb{T}$. If $f \in N$, write $f = \varphi/\psi$ as in Theorem 4.2. Then

$$f^*(\zeta) = \frac{\lim_{z \hat{\to} \zeta} \varphi(\zeta)}{\lim_{z \hat{\to} \zeta} \psi(\zeta)}$$

and $\log |f^*| = \log |\varphi^*| - \log |\psi^*| \in L^1(\mathbb{T})$. □

We obtain the following maximum principle for the class N.

Corollary 4.6. *Assume that $f, g \in N$. If $f^* = g^*$ on a set of positive Lebesgue measure. Then, $f \equiv g$.*

Proof. Let $h = f - g$. If $h \not\equiv 0$ then $\log |h^*| \in L^1(\mathbb{T})$, and thus h^* cannot vanish on a set of positive Lebesgue measure. □

We will now discuss the zeros of a function $f \in N$. Recall, that for $z_0 \in \mathbb{D}$, we defined the Blaschke factor by

$$b_{z_0}(z) := \begin{cases} -\frac{\overline{z_0}}{|z_0|} \frac{z - z_0}{1 - \overline{z_0} z} & \text{if} \quad z_0 \neq 0 \\ z & \text{if} \quad z_0 = 0. \end{cases}$$

and we showed that for $B = \prod_{j \ge 1} b_{z_j}$ it holds that $B \not\equiv 0$, if and only if z_j satisfy the Blascke condition

$$\sum_{j \ge 1} (1 - |z_j|) < \infty. \tag{4.2}$$

We will show the zeros of f satisfy the Blaschke condition. Before, we need a statement about boundary behaviors of Blaschke products.

Lemma 4.7. *Assume that* $(z_k)_{k\in\mathbb{N}}$, $z_k \in \mathbb{D}$, *satisfy the Blaschke condition* (4.5) *and let*

$$B(z) = \prod_{k\geq 1} b_{z_k}(z).$$

Then $|B^*| = 1$ *a.e. on* $\mathbb{T}$ *and*

$$\lim_{r\nearrow 1}\int_{\mathbb{T}} \log|B_r(\zeta)|d\lambda(\zeta) = 0. \tag{4.3}$$

Proof. We know already that $B \in \mathrm{Hol}(\mathbb{D})$ and $B \not\equiv 0$ and $|B| \leq 1$ in $\mathbb{D}$. Thus, B has nontangential boundary limits a.e. on $\mathbb{T}$ and $\|B^*\|_\infty \leq 1$. Since $\log|B|$ is subharmonic, the limit in (4.3) exists and clearly it is nonpositive. Fix $N \in \mathbb{N}$ and let

$$B_N(z) = \prod_{k=1}^{N} b_{z_k}(z) \quad \tilde{B}_N(z) = \prod_{k=N+1}^{\infty} b_{z_k}(z).$$

Since for $\zeta \in \mathbb{T}$, it holds that $|b_{z_k}(\zeta)| = 1$ it follows that for fixed N

$$\lim_{r\nearrow 1}\int_{\mathbb{T}} \log|B_N(\zeta r)|d\lambda(\zeta) = 0$$

and thus,

$$\lim_{r\nearrow 1}\int_{\mathbb{T}} \log|B(r\zeta)|d\lambda(\zeta) = \lim_{r\nearrow 1}\int_{\mathbb{T}} \log|\tilde{B}_N(\zeta r)|d\lambda(\zeta).$$

Hence by Theorem 3.50 and Fatou's lemma we get for fixed N

$$\begin{aligned}\log|\tilde{B}_N(0)| &= \lim_{r\searrow 0}\int_{\mathbb{T}} \log|\tilde{B}_N(\zeta r)|d\lambda(\zeta) \leq \lim_{r\nearrow 1}\int_{\mathbb{T}} \log|\tilde{B}_N(\zeta r)|d\lambda(\zeta)\\ &= \lim_{r\nearrow 1}\int_{\mathbb{T}} \log|B(r\zeta)|d\lambda(\zeta) \leq \int_{\mathbb{T}} \lim_{r\nearrow 1} \log|B(r\zeta)|d\lambda(\zeta) = \int_{\mathbb{T}} \log|B^*(\zeta)|d\lambda(\zeta) \leq 0.\end{aligned}$$

For N sufficiently large (so that $z_k \neq 0$ for $k \geq N$) we have

$$\log|\tilde{B}_N(0)| = \sum_{k=N+1}^{\infty} \log|z_k|.$$

Since this is a convergent sum, it follows that $\log|\tilde{B}_N(0)| \to 0$ as $N \to \infty$. Hence, we obtain (4.3) and $\log|B^*(\zeta)| = 0$ a.e and thus $|B^*| = 1$ a.e.. □

Theorem 4.8. *Let* $f \in N$. *Then, the zeros of* f *satisfy the Blaschke condition. Let* B *be the convergent Blaschke product, whose zeros coincide with the ones of* f *and define* $g = f/B$. *Then* g *does not vanish in* $\mathbb{D}$, $g \in N$ *and*

$$\|f\|_0 = \|g\|_0.$$

Proof. Since we can write $f = \varphi/\psi$ with analytic and bounded φ, ψ, and the zeros of f coincide with the zeros of φ, it suffices to show the first statement for $f \in \mathrm{Hol}(\mathbb{D})$ and $|f| \leq 1$ in $\mathbb{D}$. By multiplying f with an appropriate constant, we can assume that $|f| \leq 1$ on $\mathbb{D}$. Moreover, since $z^m \in N$, wlog we can assume by Corollary 4.4 that $f(0) \neq 0$.

Let z_k be the zeros of f (counting multiplicity) ordered so that $|z_k|$ is monotonically increasing. Define

$$B_N(z) = \prod_{k=1}^{N} b_{z_k}(z).$$

We want to show that

$$\left|\frac{f(z)}{B_N(z)}\right| \leq 1, \quad z \in \mathbb{D}. \tag{4.4}$$

Fix $\epsilon > 0$ and choose R sufficiently large, so that for $k \leq N$, $|z_k| < R$ and $|B_N(z)| \geq 1 - \epsilon$ for $R \leq |z| \leq 1$. It follows that for every $R \leq r < 1$

$$\left|\frac{f(z)}{B_N(z)}\right| \leq \frac{1}{1-\epsilon} \quad |z| = r,$$

and by the maximum principle this inequality holds for $|z| \leq r$. Since, ϵ was arbitrary, we obtain (4.4). Hence, for all N we have

$$0 < |f(0)| \leq |B_N(z)| = \prod_{k=1}^{N} |z_k|$$

Lemma 2.31 shows that the zeros satisfy the Blaschke condition.

It remains to show that $\|f/B\|_0 = \|f\|_0$. We have

$$\log^+ \left|\frac{f}{B}\right| \leq \log^+ |f| + \log^+ \left|\frac{1}{B}\right| = \log^+ |f| - \log |B|$$

From (4.3) it follows that $\|f/B\|_0 \leq \|f\|_0$. The other inequality is clear, since $|f| \leq |f/B|$. □

Remark 4.9. *Let us mention for later reference, that in the proof we showed that if $f \in \mathrm{Hol}(\mathbb{D})$ and $|f| \leq c$, B the Blascke producto of its zeros, then $|f/B| \leq c$.*

We are now ready to proof the main factorization theorem for the class N. We start with a definition.

Definition 4.10. *Let $f \in \mathrm{Hol}(\mathbb{D})$.*

(i) f is called inner, if $|f| \leq 1$ and $|f^*| = 1$ a.e.;

(ii) f is called singular inner, if f is inner and does not vanish in $\mathbb{D}$;

(iii) f is called outer for the class N, if it can be written as

$$f(z) = Ce^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log \phi(\zeta) d\lambda(\zeta)}, \tag{4.5}$$

where C is a unimodular constant and ϕ is a positive measurable function on $\mathbb{T}$ such that $\log \phi \in L^1(\mathbb{T})$.

If $F(z) = e^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)}$, for some complex measure μ, we will repeatedly use the following identity:

$$\operatorname{Re} \log F(z) = \log |F(z)| = \mathscr{P}[\mu](z).$$

Note that there exists an analytic log of F, since F does not vanish on $\mathbb{D}$ and $\operatorname{Re} \log F$ does not depend on the particular branch. The farcorization theorem for N reads now as follows:

THEOREM 4.11. *Let $f \in \operatorname{Hol}(\mathbb{D})$. Then $f \in N$ if and only if f can be written in the form*

$$f = CB\frac{S_1}{S_2}F, \tag{4.6}$$

where C is an unimodular constant, B is a Blaschke product, S_1, S_2 are singular inner functions and F is an outer function. B is uniquely determined by the zeros of f and if F is written as in (4.5), then $\phi = \log|f^*|$.

Conversely, every function of this form belongs to N.

PROOF. Let $f \in N$ and denote by f^* its boundary values. By Theorem 4.8 we can assume that f does not vanish in $\mathbb{D}$. Set $g = \log f$. Then $g \in \operatorname{Hol}(\mathbb{D})$ $\operatorname{Re} g = \log|f|$. Since $f \in N$ and g does not vanish in $\mathbb{D}$, it follows from Corollary 4.3 that

$$\sup_{r\in[0,1)} |\operatorname{Re} g_r(\zeta)| d\lambda(\zeta) < \infty.$$

Thus, by Corollary 3.29, there exists a complex measure μ and a real constant K so that

$$\log f(z) = g(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + iK.$$

Write $d\mu(\zeta) = \phi(\zeta)d\lambda(\zeta) + d\mu_s(\zeta)$, where $\phi \in L^1(\mathbb{T})$ and μ_s is singular with respect to λ. Moreover, decompose $\mu_s = \mu_2 - \mu_1$, where μ_i are positive finite measures still singular with respect to λ. Set

$$S_j(z) = e^{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu_j(\zeta)}$$

and

$$F(z) = e^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \phi(\zeta) d\lambda(\zeta)}.$$

Thus, we have

$$f = e^{iK} B\frac{S_1}{S_2}F.$$

Since μ_j are positive, we clearly have $|S_j| \leq 1$. Moreover, by Fatou's theorem, more precisely Corollary 3.23, we have $|S_j^*| = 1$ a.e.. Now we show that every function of the form (4.6) belongs to N. Since $|B| \leq 1, |S_j| \leq 1$ we only need to show that $F \in N$. Let $\phi \in L^1(\mathbb{T})$ and write $\phi = \phi_1 - \phi_2$, where $\phi_+ = \max\{\phi, 0\}$, $\phi_- = -\min\{\phi, 0\}$. Then we have $F = F_1/F_2$ where

$$F_1(z) = e^{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \phi_-(\zeta) d\lambda(\zeta)}, \quad F_2(z) = e^{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \phi_+(\zeta) d\lambda(\zeta)}.$$

It remains to show the uniqueness claim. The Blaschke product is uniquely determined by the zeros of f. Moreover, $|f^*| = |F^*|$ a.e. Thus, we obtain by Fatou's theorem

$$\phi(\zeta) = \lim_{z \to \zeta} \mathscr{P}[\phi](z) = \lim_{z \to \zeta} \operatorname{Re} \log F(z) = \lim_{z \to \zeta} \log|F(z)| = \log|F^*(\zeta)| = \log|f^*(\zeta)|,$$

a.e. and thus, ϕ is uniquely determined by the boundary values of $|f^*|$ and the uniqueness claim follows. □

As a corollary of the construction together with Corollary 3.30 we obtain:

COROLLARY 4.12. *Let $f \in \operatorname{Hol}(\mathbb{D})$. Then f is singular inner, if and only of there exists an unimodular constant C and a real, finite, positive measure ν on $\mathbb{T}$, which is singular with respect to λ, such that*

$$f = Ce^{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\nu(\zeta)}, \quad z \in \mathbb{D}.$$

With some abuse of notation, for a measureable function $\phi : \mathbb{T} \to \mathbb{C}$ let us define

$$\|\phi\|_0 = \int_{\mathbb{T}} \log^+ |\phi(\zeta)| d\lambda(\zeta).$$

For $f \in N$, the Lemma of Fatou implies

$$\|f\|_0 = \lim_{r \nearrow 1} \int_{\mathbb{T}} \log^+ |f(r\zeta)| d\lambda(\zeta) \geq \int_{\mathbb{T}} \lim_{r \nearrow 1} \log^+ |f(r\zeta)| d\lambda(\zeta) = \|f^*\|_0. \tag{4.7}$$

But $\|f\|_0 > \|f^*\|_0$ can happen as the following example shows:

EXAMPLE 4.13. Consider

$$f(z) = e^{\frac{1+z}{1-z}} = e^{\int \frac{\zeta+z}{\zeta-z} d\nu(\zeta)},$$

where ν is a dirac delta at $\zeta_0 = 1$. Then we have

$$\log |f(z)| = \operatorname{Re} \log f(z) = \operatorname{Re} \frac{1+z}{1-z} = \frac{1-|z|^2}{|1-z|^2} = P(1, z).$$

Hence, $\log |f|$ has boundary values 0 a.e. and we get for any $0 < r < 1$

$$\|f\|_0 \geq \int_{\mathbb{T}} \log^+ |f(r\zeta) d\lambda(\zeta) \geq \log^+ |f(0)| = 1 > 0 = \|f^*\|_0$$

Note that $f = S^{-1}$, where S is a singular inner function. ◀

The above example contains the only thing that can go wrong in (4.8). As we will see, in this sense the class N behaves essentially different to the Hardy classes. We therefore define the smaller class N_+.

DEFINITION 4.14. *The Smirnov class $N_+ \subset N$ consists of those $f \in N$ for which*

$$\|f\|_0 = \|f^*\|_0.$$

The following theorem shows, that the above division by a singular factor in the factorication (4.6) is the only reason why equality may fail.

THEOREM 4.15. *Let $f \in \operatorname{Hol}(\mathbb{D})$. Then the following are equivalent:*

(i) $f \in N_+$;

(ii) $f \in N$ and $\lim_{r \nearrow 1} \| \log^+ |f_r| - \log^+ |f^*| \|_1 = 0$;

(iii) f admits a factorization of the form $f = CBSF$, where C is a unimodular constant, B a Blaschke product, and S, F are singular inner and outer functions, respectively.

The proof of the theorem requires some preparatory work.

LEMMA 4.16. *Let $(X, \mathscr{A}, \mu)$ be a measure. For all $n \in \mathbb{N}$, let $u_n, v_n : X \to \mathbb{R}$ be measureable function, such that $v_n \in L^1(\mu)$ and*

$$|u_n| \leq v_n.$$

If $u_n \to u$, $v_n \to v$ pointwise μ-almost everywhere and $\int v_n d\mu \to \int v d\mu$, then

$$\lim_{n \to \infty} \int u_n d\mu = \int u d\mu.$$

PROOF. From the lemma of Fatou, we get

$$\int v d\mu + \int u d\mu = \int (v+u) d\mu = \int \liminf_n (v_n + u_n) d\mu$$
$$\leq \liminf \int (v_n + u_n) d\mu = \int v d\mu + \liminf \int u_n d\mu.$$

That is $\int u d\mu \leq \liminf \int u_n d\mu$. Considering $\int v d\mu - \int u d\mu$, we obtain in the same way $\limsup \int u_n d\mu \leq \int u d\mu$. Thus,

$$\limsup \int u_n d\mu \leq \int u d\mu \leq \liminf \int u_n d\mu.$$

□

COROLLARY 4.17. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $p \in (0, \infty)$ and $f_n, f \in L^p(\mu)$, $n \in \mathbb{N}$. If $f_n \to f$ pointwise μ-almost everywhere and $\|f_n\|_p \to \|f\|_p$, then*

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

PROOF. Set $u_n = |f_n - f|^p$, $u = 0$, $v_n = 2^p(|f_n| + |f|)$ and $v = 2^{p+1}|f|$. Then pointwise we have

$$|u_n|^{1/p} = |f_n - f| \leq |f_n| + |f| \leq 2\max\{|f_n|, |f|\}$$
$$= 2(\max\{|f_n|^p, |f|^p\})^{1/p} \leq 2(|f_n|^p + |f|^p)^{1/p} = v_n^{1/p}.$$

Thus, the claim follows by Lemma 4.16. □

Recall that in Theorem 3.27 we showed that u harmonic in $\mathbb{D}$ is the Poisson integral of a non-negative measure μ if and only if $u \geq 0$. The measure μ was constructed as a w^*-limit if $u_{r_n} d\lambda$. In particular

$$\|u_{r_n}\|_1 = \int_{\mathbb{T}} u_{r_n} d\lambda \to \int_{\mathbb{T}} d\mu. \tag{4.8}$$

We will apply this for $d\mu = \log^+ f^* d\lambda$.

PROOF OF THEOREM 4.15. (i) $\Longrightarrow$ (ii): This follows directly by Corollary 4.17 with $p = 1$ and the functions $\log^+ |f_r|$ and $\log^+ |f^*|$.

(iii) $\Longrightarrow$ (i): Let $f = CBSF$. Then $f \in N$. By (4.7) it requires to show that $\|f\|_0 \leq \|f^*\|_0$. Clearly $|f| \leq |F|$. Hence, $\log(|F|) = \mathscr{P}[\log|f^*|] \leq \mathscr{P}[\log^+|f^*|]$ and thus

$$\log^+ |f| \leq \mathscr{P}[\log^+ |f^*|].$$

It follows that

$$\|f_r\|_0 = \|\log^+ |f_r|\|_1 \leq \|\mathscr{P}[\log^+ |f^*|]_r\|_1.$$

By (4.8) we get for some sequence $r_n \to 1$

$$\|\mathscr{P}[\log^+ |f^*|]_{r_n}\|_1 \to \|\log^+ |f^*|\|_1 = \|f^*\|_0.$$

On the other side, the left hand side tends to $\|f_0\|$ and therefore we get $\|f\|_0 \leq \|f^*\|_0$.

(ii) $\Longrightarrow$ (iii): Since $f \in N$, we can write $f = CB\frac{S_1}{S_2}F$. Our goal is to show that $|\frac{S_1}{S_2}| \leq 1$, which implies that $\frac{S_1}{S_2}$ is an singular inner function by definition. Choose $r \in (0, 1)$ such that

f does not vanish on $r\mathbb{T}$. The function $\log|f(rz)|$ is subharmonic on $\frac{1}{r}\mathbb{D} \supset \mathbb{D}$ and takes the values $\log|f_r|$ on $\mathbb{T}$. For $z \in \mathbb{D}$ we thus have

$$\log|f(rz)| \le \mathscr{P}[\log|f_r|](z) = \mathscr{P}[\log^+|f_r|](z) - \mathscr{P}[\log^-|f_r|](z).$$

Now choose $r_n \in (0,1)$ tending to 1 such that f does not vanish on $r_n\mathbb{T}$. By the Lemma of Fatou we have

$$-\liminf_{n\to\infty} \mathscr{P}[\log^-|f_{r_n}|](z) \le -\mathscr{P}[\log^-|f^*|](z), \quad z \in \mathbb{D}.$$

Note that by assumption $\|\log^+|f_{r_n}| - \log^+|f^*|\|_1 \to 0$. Thus, for fixed $z \in \mathbb{D}$, we get

$$\begin{aligned}\left|\mathscr{P}[\log^+|f_{r_n}|](z) - \mathscr{P}[\log^+|f^*|](z)\right| &\le \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|} |\log^+|f_{r_n}(\zeta)| - \log^+|f^*(\zeta)||d\lambda(\zeta) \\ &\le \max_{\zeta\in\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|} \|\log^+|f_{r_n}| - \log^+|f^*|\|_1 \to 0.\end{aligned}$$

Since $\log|f(r_n z)| \to \log|f(z)|$ we get by combining these statements

$$\log|f(z)| \le \mathscr{P}[\log^+|f^*|](z) - \mathscr{P}[\log^-|f^*|](z) = \mathscr{P}[\log|f^*|](z) = \log|F(z)|.$$

Hence $|\frac{f}{F}| \le 1$, and by Remark 4.9 also $|\frac{S_1}{S_2}| = |\frac{f}{FB}| \le 1$. This concludes the proof. □

We have already used implicitly in the proof that for F outer we have

$$\log|F| = \mathscr{P}[\log|F^*|]$$

and for $f \in N_+$ we have

$$\log|f| = \log|BSF| \le \log|F| = \mathscr{P}[\log|f^*|].$$

The following proposition shows that this already characterizes outer functions.

PROPOSITION 4.18. *Let $f \in N_+$, $f \not\equiv 0$. Then the following are equivalent:*

(i) f is outer;

(ii) For all $z \in \mathbb{D}$

$$\log|f(z)| = \mathscr{P}[\log|f^*|](z).$$

(iii) There exists $z \in \mathbb{D}$ such that

$$\log|f(z)| = \mathscr{P}[\log|f^*|](z);$$

(iv) If $g \in N^+$ and $|f^*| = |g^*|$ a.e. on $\mathbb{T}$, then for every $z \in \mathbb{D}$

$$|g(z)| \le |f(z)|.$$

PROOF. (i) $\iff$ (ii) follows by definition and (ii) $\implies$ (iii) is clear. Also (i) $\implies$ (iv) follows immediately from the inner outer factorization. We show (iii) $\implies$ (i). Let F be the outer function determined by $|f^*|$. Then it always holds $|f| \le |F|$, i.e.,

$$\left|\frac{f}{F}\right| \le 1.$$

By assumption, $\left|\frac{f}{F}\right|$ has a local maximum and thus $f = CF$ and hence f is outer.

(iv) $\implies$ (i) Let g be the outer function determined by the boundary values of f. Then $|f| \leq |g|$ holds always and by assumption $|f| = |g|$, i.e.,

$$\left|\frac{f}{g}\right| = 1.$$

That is $f = Cg$ and hence f is outer. □

As a corollary we obtain the following strong maximum principle:

COROLLARY 4.19. *Let $f \in N_+$. Then f is inner if and only if $|f^*| = 1$ a.e.*

PROOF. Write $f = AF$ where A is inner and F is outer. We need to show that $|f| \leq 1$ in $\mathbb{D}$. We have

$$\log |f| \leq \log |F| = \mathscr{P}[\log |f^*|] = 0.$$

□

Note that this does not hold if f is merely of bounded type.

4.1.1. Hardy spaces. Recall that for $f \in \mathrm{Hol}(\mathbb{D})$, we defined $f_r(\zeta) = f(r\zeta)$.

DEFINITION 4.20. *Let $p \in (0, \infty]$. We define the Hardy space of the unit disc by*

$$H^p = H^p(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) \mid \sup_{r\in[0,1)} \|f_r\|_p\} < \infty$$

For $f \in H^p$ we define

$$\|f\|_p = \sup_{r\in[0,1)} \|f_r\|_p.$$

By Theorem 3.50 $\|f_r\|_p$ are monotonic and we have in fact

$$\sup_{r\in[0,1)} \|f_r\|_p = \lim_{r\nearrow 1} \|f_r\|_p.$$

Moreover, by Hölder's inequality, we have for $0 < p \leq p' \leq \infty$ that $\|f\|_p \leq \|f\|_p'$ and thus $H^{p'} \subseteq H^p$.

We will need the following elementary inequality:

LEMMA 4.21. *Let $0 < p \leq 1$ and $x, y \geq 0$. Then*

$$\left|\log^+ x - \log^+ y\right| \leq \frac{1}{p}|x-y|^p.$$

PROOF. Consider on $[1, \infty)$ the functions $f(t) = t^p - 1$ and $g(t) = (t-1)^p$. Then $f(1) = g(1) = 0$ and since $p - 1 \leq 0$

$$f'(t) = pt^{p-1} \leq p(t-1)^{p-1} = g'(t), \quad t \geq 1.$$

Thus $f \leq g$ on $[1, \infty)$. From $1 + x \leq e^x$ it follows that $\log(t^p) \leq f(t)$ and thus

$$\log t \leq \frac{1}{p}(t-1)^p, \quad t \geq 1.$$

We now get the claim by a case distinction.

If $1 \leq y \leq x$, we have

$$|\log^+ x - \log^+ y| = \log \frac{x}{y} \leq \frac{1}{p}\left(\frac{x}{y} - 1\right)^p \leq \frac{1}{p}\left(\frac{x}{y} - 1\right)^p y^p = \frac{1}{p}|x-y|^p.$$

If $y \le 1 \le x$ we have

$$|\log^+ x - \log^+ y| = \log x \le \frac{1}{p}(x-1)^p \le \frac{1}{p}(x-y)^p = \frac{1}{p}|x-y|^p.$$

If $y \le x \le 1$ we have $|\log^+ x - \log^+ y| = 0$ and the claim is trivial. □

Choosing above $y = 0$ it follows that $\log^+ x \le \frac{1}{p}|x|^p$ and thus we see that $H^p \subset N$ for all $p > 0$. We thus immediately get:

COROLLARY 4.22. *Let $p \in (0, \infty]$. Then $H^p \subset N$. In particular every $f \in H^p$ has nontangential boundary values a.e.. That is for a.e. $\zeta \in \mathbb{T}$ it holds that*

$$f^*(\zeta) := \lim_{z \hat{\to} \zeta} f(z).$$

Moreover, the zeros of f satisfy the Blaschke condition.

As for the class N, we can separate the zeros without affecting $\|\cdot\|_p$.

LEMMA 4.23. *Let $p \in (0, \infty]$, $f \in H^p$ and B the Blaschke product formed by the zeros of f. Then $\frac{f}{B} \in H^p$ and $\left\|\frac{f}{B}\right\|_p = \|f\|_p$.*

PROOF. For the case $p = \infty$ it follows from Remark 4.9. Consider now $p \in (0, \infty)$. Let again B_N denote the finite Blaschke product of the first N zeros of f (orderd by $|z_1| \le \cdots \le |z_N|$). Since $\lim_{r \nearrow 1} B_N(r\zeta)$ uniformly on $\mathbb{T}$, we obtain by subharmonicity of $|f/B_N|^p$

$$\left\|\left(\frac{f}{B_N}\right)_r\right\|_p \le \lim_{r \nearrow 1}\left\|\left(\frac{f}{B_N}\right)_r\right\|_p = \lim_{r \nearrow 1}\|f_r\|_p = \|f\|_p.$$

Since $B_N \to B$ locally uniformly in $\mathbb{D}$ we see that for fixed $r < 1$

$$\lim_{N\to\infty}\left\|\left(\frac{f}{B_N}\right)_r\right\|_p = \left\|\left(\frac{f}{B}\right)_r\right\|_p.$$

Thus, $\frac{f}{B} \in H^p$ and $\left\|\frac{f}{B}\right\|_p \le \|f\|_p$. The other inequality is trivial since $|B| \le 1$. □

The theorem below explains a posteriori the definiton of N_+.

THEOREM 4.24. *Let $p \in (0, \infty)$ and $f \in H^p$. Then $\|f^*\|_p = \|f\|_p$ and*

$$\lim_{r \nearrow 1}\|f_r - f^*\|_p = 0.$$

PROOF. For $p \in (1, \infty)$ this follows from Theorem 3.27(ii). Let now $p \in (0, \infty)$ arbitrary and $f \in H^p$. Let B be the Blaschke product of zeros of f and $g = \frac{f}{B}$. Then $g \in H^p$ and $\|g\|_p = \|f\|_p$. Since g has no zeros in $\mathbb{D}$, there exists analytic h with $h^{\frac{2}{p}} = g$. Since $\|h_r\|_2 = \|g_r\|_p$, we have $h \in H^2$. Moreover, $|h^*|^{2/p} = |g^*| = |f^*|$, and in $\mathbb{D}$ we have $|f(z)| \le |g(z)|$. Hence, by the Lemma of Fatou it follows that

$$\begin{aligned}\int_{\mathbb{T}} |f_r|^p d\lambda &\le \int_{\mathbb{T}} |g_r|^p d\lambda = \int_{\mathbb{T}} |h_r|^2 d\lambda \le \int_{\mathbb{T}} |h^*|^2 d\lambda \\ &= \int_{\mathbb{T}} |g^*|^p d\lambda = \int_{\mathbb{T}} |f^*|^p d\lambda \le \liminf_{r \nearrow 1}\int_{\mathbb{T}} |f_r| d\lambda = \|f\|_p.\end{aligned}$$

Note that for the second inequality, we used that $\int_{\mathbb{T}} |h_r|^2 d\lambda \leq \|h\|_2^2 = \|h^*\|_2^2$. Sending $r \to 1$ in this relation shows that $\|f^*\|_p = \|f\|_p$. It now follows from Corollary 4.17 that $\lim_{r\nearrow 1} \|f_r - f^*\|_p = 0$. □

Remark 4.25. *The claim is not correct for H^∞. Otherwise, every bounded analytic function in $\mathbb{D}$ would have a continuous extension to $\mathbb{D}$. But this is not the case, which can be seen by basically any inner function. Take e.g.*

$$f(z) = e^{-\frac{1+z}{1-z}}.$$

Note that still

$$\lim_{r\nearrow} \|f_r\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| = \|f^*\|_\infty.$$

For $\|f\|_\infty \leq \|f^*\|_\infty$ holds by subharmonicity, and if $\sup_{z\in\mathbb{D}} |f(z)| < C$, then $|f^*| < C$ and hence $\|f\|_\infty = \|f^*\|_\infty$.

If $p \in (1, \infty]$, then $\|\cdot\|$ defines a norm on H^p. If $p \in (0,1)$, $d_{H^p}(f,g) = \|f-g\|_p^p$ is a metric. Recall also that similar norms and metrices are defined on L^p.

Corollary 4.26. *For any $p \in (0, \infty]$, the map*

$$\begin{cases} H^p(\mathbb{D}) & \to & L^p(\mathbb{T}) \\ f & \mapsto & f^* \end{cases} \tag{4.9}$$

defines an isometry. If $f \in H^1$, then f can be recovered from its boundary values as an Poisson integral

$$f(z) = \int_{\mathbb{T}} P(z,\zeta) f^*(\zeta) d\lambda(\zeta)$$

or by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} d\zeta.$$

Proof. It only requires to proof the integral representations. If $f \in H^1$, then $\|f_r - f^*\|_1 \to 0$. Now for every $r < 1$ we have

$$f_r(z) = \int_{\mathbb{T}} P(z,\zeta) f_r(\zeta) d\lambda(\zeta).$$

For fixed $z \in \mathbb{D}$, as we send $r \to 1$, the left-hand side tends to $f(z)$ and the right-hand side to $\int_{\mathbb{T}} P(z,\zeta) f(\zeta) d\lambda(\zeta)$. The representation as Cauchy integral is proved in the same way, noting that

$$\frac{1}{2\pi i} \frac{d\zeta}{\zeta} = d\lambda(\zeta).$$

□

As an immediate corollary we obtain the following result to F. and M. Riez:

Corollary 4.27. *Let μ be a complex measure such that for every $n = 1, 2, 3, \ldots$*

$$\int_{\mathbb{T}} \zeta^n d\mu(\zeta) = 0,$$

then μ is absolutely continuous.

Proof. If μ is a complex measure, then the harmonic function $u = \mathscr{P}[\mu]$ satisfies

$$\sup_{r\in[0,1)} \|u_r\|_1 < \infty.$$

Moreover, writing $P(z,\zeta)$ as a power series we get

$$u(z) = \sum_{j=1}^{\infty} c_{-j}\overline{z}^j + \sum_{j=0}^{\infty} c_j z^j,$$

where

$$c_j = \int_{\mathbb{T}} \zeta^{-j} d\mu(\zeta)$$

denote the moments of μ. Thus, our assumption shows that u is analytic and thus belongs to H^1. By Corollary 4.26 we have

$$u = \mathscr{P}[u^*] = \mathscr{P}[\mu].$$

By uniqueness of the measure Theorem 3.27 we get that $d\mu = u^* d\lambda$. □

Corollary 4.28. *For any* $p \in (0,\infty]$, $H^p \subset N_+$.

Proof. Let $1 > p > 0$. If $\|f_r - f^*\|_p \to 0$, then by Lemma 4.21 $\|\log^+|f_r| - \log^+|f^*|\|_1 \to 0$ and thus $f \in N^+$ by Theorem 4.15. □

We conclude this section by describing the factorization for H^p functions. Recall that an out function for N_+ was of the form

$$Ce^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|f^*(\zeta)| d\lambda(\zeta)}, \quad |C| = 1,$$

where $\log|f^*(\zeta)| \in L^1$. A similar definition holds for H^p.

Definition 4.29. *Let* $p \in (0,\infty]$. *A function* f *is called outer for* H^p, *if it is of the form*

$$Ce^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|\varphi(\zeta)| d\lambda(\zeta)}, \quad |C| = 1, \tag{4.10}$$

such that $\log|\varphi| \in L^1$ *and* $\varphi \in L^p$.

Proposition 4.30. *Let* $p \in (0,\infty]$. *Then* $f \in H^p$ *if and only if* f *can be represented in the form*

$$f = CBSF,$$

where C *is an unimodular constant,* B *a Blaschke product,* S *a singular inner function and* F *outer for* H^p. *If* $f \in H^p$ *is given, then in the representation of the outer function in* (4.10) *one has* $\varphi = f^*$.

Proof. We first show that a function of this form belongs to H^p. Since a product of a bounded function and an H^p function belongs to H^p we only need to show that an outer function for H^p, belongs to H^p. Let F be outer for H^p. Then we have

$$\log|F(z)| = \int_{\mathbb{T}} \log|\phi(\zeta)| P(z,\zeta) d\lambda(\zeta).$$

Multiplying this identity with p and applying Jensen's inequality for exp and the probability measure $P(z,\zeta)d\lambda(\zeta)$, we get

$$|F(z)|^p = \exp\left(\int_{\mathbb{T}} \log|\phi(\zeta)|^p (P(z,\zeta)d\lambda(\zeta))\right) \leq \int_{\mathbb{T}} |\phi(\zeta)|^p P(z,\zeta)d\lambda(\zeta) = \mathscr{P}[|\phi(\zeta)|^p](z). \tag{4.11}$$

Now let $p \in (0,\infty)$. Since $|\phi(\zeta)|^p \in L^1$, $|\phi(\zeta)|^p d\lambda$ defines a complex measure and hence,

$$\sup_{r\in[0,1)} \|\mathscr{P}[|\phi(\zeta)|^p]_r\|_1 < \infty.$$

In particular, since $\||F_r|^p\|_1 = \|F_r\|_p$ we get

$$\sup_{r\in[0,1)} \|F_r\|_p < \infty.$$

and since F is clearly analytic, we have $F \in H^p$.

If $p = \infty$, we can still apply (4.11) to see

$$|F(z)| \leq \mathscr{P}[|\phi(\zeta)|](z)$$

with $\phi \in L^\infty$. The claim now follows by the same arguments using Theorem 3.27(iii).

If $f \in H^p$, then $f \in N_+$ and we get

$$f = CBSF,$$

where

$$F(z) = e^{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|f^*(\zeta)| d\lambda(\zeta)}.$$

Since $f^* \in L^p$, the claim follows. □

We get one more important corollary from this construction.

Corollary 4.31. *If $f \in N_+$ and $f^* \in L^p$, then $f \in H^p$.*

This corollary, once again shows the importance of the Smirnov class. If one knows a priori that $f \in N_+$, it suffices to study f^* to check if $f \in H^p$.

We now show that H^p is complete. This requires a lemma:

Lemma 4.32. *Let $p \in (0,\infty)$ and $f \in H^p$. Then for every $z \in \mathbb{D}$*

$$|f(z)| \leq 2^{\frac{1}{p}} \|f\|_p \frac{1}{(1-|z|)^p}.$$

Proof. Recall that

$$P(z,\zeta) \leq \frac{1+|z|}{1-|z|} \leq \frac{2}{1-|z|}.$$

Write $f = CBSF$. Then we get as above,

$$|f(z)|^p \leq |F(z)|^p \leq \int_{\mathbb{T}} P(z,\zeta)|f^*(\zeta)|^p d\lambda(\zeta) \leq \frac{2}{1-|z|}\|f\|_p^p$$

and the claim follows. □

Theorem 4.33. *For every $p \in (0,\infty]$, the space H^p is complete.*

PROOF. First note that if f_n converges in H^p, then f_n converges locally uniformly in $\mathbb{D}$. For $p = \infty$ this is trivial and for $p \in (0, \infty)$ this follows from Lemma 4.32. Assume now that $(f_n)_{n\in\mathbb{N}}$, $f_n \in H^p$ is a Cauchy sequence in H^p. Then, it is a Cauchy sequence with respect to local uniform convergence, and thus $\lim_{n\to\infty} f_n = f \in \mathrm{Hol}(\mathbb{D})$ locally uniformly. We need to show that $f \in H^p$ and $\|f_n - f\|_p \to 0$. We restrict us to the case $p \in [1, \infty]$. For $p < 1$, one has to add the exponent p in the estimates below. Fix $\epsilon > 0$. Then we find n_0, such that for $n, m \geq n_0$ we have $\|f_n - f_m\|_p \leq \epsilon$. Fix $r < 1$ and choose $n \geq n_0$, such that $\|f_r - (f_n)_r\|_p \leq \epsilon$, which is possible due to local uniform convergence. Then we get

$$\|f_r\|_p \leq \|f_r - (f_n)_r\|_p + \|(f_n)_r - (f_{n_0})_r\|_p + \|(f_{n_0})_r\|_p \leq \epsilon + \|f_n - f_{n_0}\|_p + \|f_{n_0}\|_p \leq 2\epsilon + \|f_{n_0}\|_p$$

and thus, $f \in H^p$. Moreover,

$$\|f_r - (f_{n_0})_r\|_p \leq \|f_r - (f_n)_r\|_p + \|(f_n)_r - (f_{n_0})_r\|_p \leq 2\epsilon,$$

which shows that $\|f - f_n\|_p \to 0$. □

PROPOSITION 4.34. *For $p \in (0, \infty)$ the set of polynomials with complex coefficients are dense in H^p.*

PROOF. Let $f \in H^p$. Then $\|f_r - f^*\|_p \to 0$. Let

$$\sum_{k=0}^{\infty} a_k z^k$$

denote the Taylor series of f around zero, which converges locally uniformly on $\mathbb{D}$ to f. Fix $\epsilon > 0$. Then we can choose $r < 1$ such that

$$\|f_r - f^*\|_p < \epsilon.$$

Moreover, we find n_0 such that

$$\left\| \sum_{k=0}^{n_0} a_k r^k z^k - f_r \right\|_p = \left\| \Big(\sum_{k=0}^{n_0} a_k z^k \Big)_r - f_r \right\|_p < \epsilon.$$

Set $q(z) = \sum_{k=0}^{n_0} a_k r^k z^k \in H^p$. Then we have

$$\|q - f\|_p = \|q^* - f^*\|_p = \left\| \Big(\sum_{k=0}^{n_0} a_k \zeta^k \Big)_r - f^* \right\|_p < 2\epsilon.$$

This finishes the proof. □

This statement fails for H^∞. In this case one has that the H^∞ closure of the polynomials is the set of analytic functions with continuous extension to $\mathbb{D}$. But since again $S(z) = e^{-\frac{1+z}{1-z}} \in H^\infty$ does not have a continuous extension, we see that this is a strict subset.

4.1.2. Some descriptions of H^p. We start with giving a description of H^p that is conformal invariant. Note that integration along curves does not have this property, since by the integration rule for push forward measures, one has to take the derivative of the conformal map into account. We have seen that if $f \in \mathrm{Hol}(\mathbb{D})$ is analytic then $|f|^p$ is subharmonic. This means that in each disc of radius $r < 1$, $|f|^p$ is dominated by some harmonic function (in fact the Poisson integral of its values on the boundary of this disc).

We say that $|f|^p$ has a harmonic majorant in $\mathbb{D}$, if there exists a single harmonic function in $\mathbb{D}$ such that

$$|f(z)|^p \leq h(z), \quad z \in \mathbb{D}.$$

PROPOSITION 4.35. *Let $f \in \mathrm{Hol}(\mathbb{D})$. Then $f \in H^p$ if and only if $|f|^p$ has a harmonic majorant. The least harmonic majorant is given by*

$$h = \mathscr{P}[|f^*(\zeta)|^p].$$

In particular, $\|f\|_p = \mathscr{P}[|f^*(\zeta)|^p](0)$.

PROOF. If $|f|^p$ has a harmonic majorant h, then by the mean value theorem

$$\int_{\mathbb{T}} |f(r\zeta)|^p d\lambda(\zeta) \leq \int_{\mathbb{T}} h(r\zeta) d\lambda(\zeta) = h(0).$$

Let $f \in H^p$ and write it in its inner outer factorization $f = cBSF$. Then we have

$$|f(z)|^p \leq |F(z)|^p \leq \mathscr{P}[|f^*|^p](z),$$

which is harmonic. $\|f\|_p = \mathscr{P}[|f^*(\zeta)|^p](0)$ is clear. We need to show that $h = \mathscr{P}[|f^*(\zeta)|^p]$ is the least harmonic majorant. That is, if u is another harmonic majorant, then for all $z \in \mathbb{D}$, $h(z) \leq u(z)$. Indeed, for any $r < 1$ we have

$$\int_{\mathbb{T}} |f(r\zeta)|^p P(z,\zeta) d\lambda(\zeta) \leq \int_{\mathbb{T}} u(r\zeta) P(z,\zeta) d\lambda(\zeta) = u(rz).$$

By the Lemma of Fatou we have

$$\int_{\mathbb{T}} |f(\zeta)^*|^p P(z,\zeta) d\lambda(\zeta) \leq \liminf_r \int_{\mathbb{T}} |f(r\zeta)|^p P(z,\zeta) d\lambda(\zeta) \leq \liminf_r u(rz) = u(z).$$

□

4.1.2.1. *Vanishing Fourier coefficients.* For $\phi \in L^1(\mathbb{T}, \lambda)$ define its Fourier coefficients by

$$\hat{\phi}(n) = \int_{\mathbb{T}} \phi(\zeta) \zeta^{-n} d\lambda(\zeta).$$

For $1 \leq p \leq \infty$, H^p can also be described in terms of the Fourier coefficients of its boundary function.

PROPOSITION 4.36. *Let $1 \leq p \leq \infty$, then $f^* \in L^p$ is the boundary function of some H^p function, if and only if the Fourier coefficients of $\widehat{f^*}(n) = 0$ for $n < 0$, i.e.,*

$$H^p = \{f^* \in L^p \mid \widehat{f^*}(n) = 0 \text{ for all } n < 0\}.$$

If $f \in H^p$, then

$$f(z) = \sum_{n=0}^{\infty} \widehat{f^*}(n) z^n.$$

PROOF. Write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then compute

$$\int_{\mathbb{T}} f(r\zeta) \zeta^{-n} d\lambda(\zeta)$$

to find

$$\widehat{f_r}(n) = \begin{cases} r^n a_n, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

On the other hand

$$|\widehat{f_r}(n) - \widehat{f^*}(n)| \leq \int |f_r(\zeta) - f^*(\zeta)| d\lambda(\zeta) = \|f_r - f^*\|_1 \to 0.$$

This implies that

$$\widehat{f^*}(n) = \begin{cases} a_n, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

Conversely, if $f \in L^p$ and $\hat{f}(n) = 0$ for $n < 0$. Set $g = \mathscr{P}[f]$. Then $\sup_{r \in (0,1)} \|g_r\|_p < \infty$ and the same computation as in Corollary 4.27 shows that $g \in \mathrm{Hol}(\mathbb{D})$. Thus, $g \in H^p$ and by Fatou's theorem $g^* = f$. □

4.1.3. Multiplication operator in H^2. The space H^2 can equipped with a scalar product and thus be turned into an Hilbert space. There are several equivalent ways to do this. The first way approach is via its Taylor (aka Fourier coefficients). Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

the we define

$$\langle f, g \rangle_{H^2} = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

With this definition it is not immediate obvious that $\langle f, f \rangle_{H^2} = \|f\|_{H^2}^2$. Equivalently, we can define

$$\langle f, g \rangle_{H^2} = \langle f, g \rangle_{H^2} = \langle f^*, g^* \rangle_{L^2(\mathbb{T})}$$

and then, by Plancherel's theorem we have

$$\langle f^*, g^* \rangle_{L^2(\mathbb{T})} = \sum_{k=1}^{\infty} \widehat{f^*}(k) \widehat{g^*}(k).$$

Since $\widehat{f^*}(k) = a_k$, we see that these two definitions coincide. It is now also clear that this scalar product induces the standard norm on H^2.

In the introduction, we have already seen that multiplication by z in H^2 is unitarily equivalent with the shift operator on ℓ^2, via the mapping

$$(a_n)_{n \in \mathbb{N}_0} \mapsto \left(z \mapsto \sum_{k=0}^{\infty} a_k z^k \right)$$

Let $\mathscr{H}$ be a Hilbert space, Y a subspace of $\mathscr{H}$ and T a linear operator on $\mathscr{H}$. Y is called invariant for T, if $T(Y) \subset Y$. In the analysis of operators it is of fundamental interest to study invariant subspaces. That is One may think of the span of eigenvectors as very simple invariant subspaces, which, in case they for an orthonormal basis, bring the operator into a very simple form. Also triangular transformation are biuld of invariant blocks of the operator. Beurling's theorem characterizes all invariant subspaces of H^2 of the multiplication operator

$$S := \begin{cases} H^2 & \to & H^2 \\ f(z) & \mapsto & zf(z) \end{cases}$$

If φ is inner, then $S_\varphi : f \mapsto \varphi f$ is an isometry on H^2.

Note that if $f \sim (a_0, a_1, a_2, \cdot)$, then $zf \sim (0, a_0, a_1, a_2, \cdot)$. Thus, the space $Y = \{f \in H^2 \mid f(0) = 0\}$ is S invariant. More generally, one can see that if $\alpha_1, \cdot, \alpha_n \in \mathbb{D}$, and $Y = \{f \in H^2 \mid f(\alpha_1) = \cdot f(\alpha_n) = 0\}$ is S invariant. The inner-outer factorization for H^2 functions shows that $Y = BH^2$, where B is the finite Blascke product with zeros at the $\alpha_k' s$. It is now reasonable (and not hard to check), that φH^2 is S invariant for any inner function φ. A much deeper result, is the Beurlin's theorem, which states that all S invariant subspaces are of this form.

THEOREM 4.37 (Beurlin's theorem). *(i) For each inner function φ, the space*

$$\varphi H^2 = \{\varphi f \mid f \in H^2\};$$

is a closed S-invariant subspace of H^2.

(ii) If φ_1, φ_1 are inner. Then $\varphi_1 H^2 = \varphi_2 H^2$ if and only if $\frac{\varphi_1}{\varphi_2}$ is constant;

(iii) Every closed S-invariant subspace Y of H^2 is of the form φH^2, for some inner function φ.

PROOF. (i): Since $S(S_\varphi H^2) = S_\varphi(SH^2) \subset S_\varphi(H^2)$, it is clear that $Y = \varphi H^2 = S_\varphi(H^2)$ is S-invariant. That it is closed follows from the fact that S_φ is an isometry.

(ii): If $\varphi_1 = C\varphi_2$, then clearly $\varphi_1 H^2 = \varphi_2 H^2$. Conversely, if $\varphi_1 H^2 = \varphi_2 H^2$, since $1 \in H^2$, we find that $\psi = \varphi_1/\varphi_2 \in H^2 \subset N_+$. Since $|\psi| = 1$ a.e. on $\mathbb{T}$, Corollary 4.19 implies that ψ is inner. Similarly $1/\psi = \varphi_2/\varphi_1$ is inner. Thus, φ is constant, which proves the claim.

(iii): Let Y be a closed S-invariant subset of H^2. Let k be the smallest number such that there exists $f \in Y$ with

$$f(z) = \sum_{j=k}^{\infty} a_j z^j, \quad a_j \neq 0.$$

By construction $f \notin zY$ and hence, zY is a closed invariant subspace of Y such that $Y \ominus zY \neq \emptyset$. Choose $\varphi \in Y \ominus zY \neq$ with $\|\varphi\|_{H^2} = 1$. Since Y is S invariant and $\varphi \in Y$, $z^n\varphi \in zY$ for all $n \geq 1$ and thus $\varphi \perp z^n \varphi$ for all $n \geq 1$. That is,

$$0 = \int_{\mathbb{T}} \varphi^*(\zeta)\overline{z^n\varphi^*(\zeta)} d\lambda(\zeta) = \int_{\mathbb{T}} |\varphi^*(\zeta)|^2 \zeta^{-n} d\lambda(\zeta), \quad n \geq 0.$$

Conjugating this, shows the same relation for for $n \leq -1$. Note that $|\varphi^*(\zeta)|^2 \in L^1$. We have just shown that

$$\forall n \in \mathbb{Z} \setminus \{0\} : \ \widehat{|\varphi^*(\zeta)|^2}(n) = 0,$$

i.e. $|\varphi^*(\zeta)|^2$ is constant a.e. on $\mathbb{T}$ and since $\|\varphi\|_{H^2} = 1$, we see that $|\varphi^*(\zeta)|^2 = 1$ a.e. in $\mathbb{T}$. Since $\varphi \in N_+$, we conclude again by Corollary 4.19 that φ is inner.

Y is a closed subspace of H^2, which contains φ and is S-invariant. Thus $\varphi p \in Y$ for any polynomial p. Since the polynomials are dense in H^2, we conclude that $\varphi H^2 \subset Y$. Assume that $h \in Y \ominus \varphi H^2$. Therefore $h \perp z^n\varphi$ for all $n \in \mathbb{N}_0$. Thus

$$0 = \int_{\mathbb{T}} h^*(\zeta) z^{-n} \overline{\varphi^*(\zeta)} d\lambda(\zeta), \quad n \geq 0.$$

We have $\varphi \in Y \ominus zY$ and $z^n h \in zY$ for all $n \geq 1$. Hence,

$$0 = \int_{\mathbb{T}} h^*(\zeta) z^{n} \overline{\varphi^*(\zeta)} d\lambda(\zeta), \quad n \geq 1.$$

Since $h^*\overline{\varphi^*} \in L^1$ and we have just shown that $\widehat{h^*\overline{\varphi^*}}(n) = 0$ for all $n \in \mathbb{N}$, we conclude that $h^*\overline{\varphi^*} = 0$ a.e.. Since φ is inner, we conclude by Corollary 4.6 that $h = 0$. □

Given $f \in H^2$, the smallest S invariant subspace of H^2 that contains f is given by

$$E_f := \overline{\operatorname{span}\{z^n f \mid n \in \mathbb{N}_0\}}^{H^2}.$$

It is a natural question to ask, if $E_f = H^2$. It turns out that this is the case precisely when f is outer.

COROLLARY 4.38. *Let $f \in H^2$ and $f = BSF$ its inner-outer factorication. Then*

$$E_f = BSH^2.$$

PROOF. Since $f = BSF \in BSH^2$ and BSH^2 is an S-invariant subspace of H^2 we conclude $E_f \subset BSH^2$. It remains to show the other inclusion.

We have $E_f = \varphi H^2$ for some inner function φ. Since $f \in E_f$, there exists $h \in H^2$ such that $f = \varphi h$. Write $h = B_1 S_1 H$ in its inner-outer factorization. Since $|f^*| = |h^*|$ there exists an unimodular constant C_1 such that $F = C_1 H$. Now we have

$$BSF = f = \varphi h = C_1 \varphi B_1 S_1 F$$

and hence $BS = C_1\varphi B_1 S_1$. Let $g \in H^2$, then $BSg = C_1 \varphi B_1 S_1 g = \varphi C_1 B_1 S_1 g \in \varphi H^2$. This shows the other inclusion

$$BSH^2 \subset \varphi H^2.$$

□

We say an inner function φ is a divisor of an inner function ψ if and only if ψ/φ is an inner function. That is,

$$\varphi | \psi :\iff \frac{\psi}{\varphi} \text{ is inner}$$

This operation corresponds exactly to the order relation of S invariant subspaces.

COROLLARY 4.39. *Let φ, ψ be inner. Then $\varphi|\psi \iff \psi H^2 \subset \varphi H^2$.*

Recall that S^* acts on H^2 by

$$f \mapsto \frac{f(z) - f(0)}{z}.$$

It is now easy to also characterize the S^* invariant subspaces.

THEOREM 4.40. *K is an S^* invariant subspaces of H^2 is and only if there exists an inner funcion φ such that*

$$K = H^2 \ominus \varphi H^2 := \mathscr{K}_\varphi.$$

PROOF. Let K be S^* invariant. For every $f \in K$ and $g \in K^\perp$ we have

$$0 = \langle S^* f, g\rangle = \langle f, Sg\rangle.$$

Thus, $K^\perp$ is S invariant and hence by Beurling's theorem we find φ such that $K = \mathscr{K}_\varphi$. □

It is not directly obvious how function in $\mathscr{K}_\varphi$ look.

LEMMA 4.41. *Let φ be inner. Then $f \in H^2$ belongs to $\mathscr{K}_\varphi$ if and only if there exists $g \in H^2$ such that*

$$f = \overline{zg}\varphi.$$

PROOF. Let $f \in H^2$. Then

$$\begin{aligned} f \in \mathscr{K}_\varphi &\iff \langle f, \varphi g\rangle = 0, \quad \forall g \in H^2 \\ &\iff \langle \overline{\varphi} f, g\rangle = 0, \quad \forall g \in H^2 \\ &\iff \overline{\varphi} f \in \overline{SH^2} \\ &\iff f \in S_\varphi \overline{SH^2}. \end{aligned}$$

□

Recall that a Hilbert space $\mathscr{H}$ of functions on $\mathbb{D}$ is called a reproducing kernel Hilbert space, if for every $w \in \mathbb{D}$, the point evaluation functional is continuous. The reproducing kernel is then the function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that

(i) $\forall w \in \mathbb{D} : K(\cdot, w) \in \mathscr{H}$;
(ii) $\forall w \in \mathbb{D}, f \in \mathscr{H} : \langle f, K(\cdot, w)\rangle = f(w)$.

We will also use the convenient notation $k_w(z) := K(z, w)$. We have already computed that the reproducing kernel for H^2 is given by

$$K(z, w) = \frac{1}{1 - z\overline{w}}.$$

LEMMA 4.42. *Let φ be inner. Then the reproducing kernel for $\mathscr{K}_\varphi$ is given by*

$$K^\varphi(z, w) = \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z}.$$

PROOF. First we show that $\frac{1-\overline{\varphi(w)}\varphi(z)}{1-\overline{w}z} \in \mathscr{K}_\varphi$. Since φ is inner, it clearly belongs to H^2. Now let $f \in H^2$. Then

$$\langle \varphi f, \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z}\rangle = \varphi(w) f(w) - \varphi(w)\langle \varphi f, \frac{\varphi(z)}{1 - \overline{w}z}\rangle = 0.$$

Now let $f \in \mathscr{K}_\varphi$. Then we get

$$\langle f, \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z}\rangle = f(w) - \varphi(w)\langle f, \frac{\varphi(z)}{1 - \overline{w}z} = f(z),$$

where we used in the last step that $\frac{\varphi(z)}{1-\overline{w}z} \in \varphi H^2$. □

LEMMA 4.43. *Let $z_0 \in \mathbb{D}$. Then*

$$\mathscr{K}_{b_{z_0}} = \operatorname{span}\{K_{z_0}\}.$$

PROOF. We have

$$b_{z_0}H^2 = \{f \in H^2 \mid f(z_0) = 0\} = \{f \in H^2 \mid \langle f, K_{z_0}\rangle = 0\} = \{K_{z_0}\}^{\perp}.$$

Taking orthogonal complements proves the claim. □

This allows us to find orthonormal basis for H^2. Let

$$k_{z_0}(z) = \frac{K(z, z_0)}{\|K(\cdot, z_0)\|} = \frac{K(z, z_0)}{\sqrt{K(\cdot, z_0)}}.$$

PROPOSITION 4.44. *Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of points such that*

$$\sum_{k\in\mathbb{N}}(1 - |z_k|) = \infty.$$

Let B_n be the finite Blaschke product with zeros at $z_1, \cdot, z_n$ and $B_0 = 1$. Define for $n \geq 0$

$$f_n = B_n k_{z_{n+1}}.$$

Then f_n is an orthonormal basis for H^2.

PROOF. We have

$$H^2 \ominus b_{z_1}H^2 = \operatorname{span}\{k_{z_1}\}.$$

Iterating this procedere, shows that (f_n) form an orthonormal system in H^2. Assume that $g \in H^2$ such that for all $n \in \mathbb{N}$

$$\langle g, f_n\rangle = 0$$

Then g has zeros at all z_k. Since on the other hand, the zeros of g satisfy the Blaschke condition $g \equiv 0$. □

Since $K(z, 0) = 1$, the choice $z_k \equiv 0$ yields the standard basis

$$1, z, z^2, \cdots .$$

Let $f_n = z^n$. Moreover, let $h(z) = z + z^{-1}$ and $T = Ph$, where $P : L^2 \to H^2$ denote the orthogonal projection. Then in the basis f_n, T has the matrix representation

$$J_0 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

If $T_n(\cos\theta) = \cos(n\theta)$ are the classical Chebyshev polynomials, then

$$zP_n(z) = \frac{1}{2}(P_{n-1}(z) + P_{n+1}(z)).$$

That, is if we define $V(x) = (P_0(x), P_1(x), \dots)^{\intercal}$, then formally

$$\frac{1}{2}J_0V(z) = zV(z).$$

This is another instance, where functional models of Hardy spaces can be used to study spectral properties of certain operators on ℓ^2.

APPENDIX A

Complex Analysis

A triangle is a contour γ composed of three line segments $[z_0 z_1 z_2 z_0]$. The inside of the triangle, denoted $\text{ins}(\gamma)$, is the the convex hull of $\{z_0, z_1, z_2\}$, i.e.,

$$\text{ins}(\gamma) = \{t_0 z_0 + t_1 z_1 + t_2 z_2 : t_0, t_1, t_2 \geq 0, t_0 + t_1 + t_2 = 1\}.$$

THEOREM A.1 (Goursat's Lemma). *Let Ω be open and $f \in \text{Hol}(\Omega)$ and γ be a triangle such that $\text{ins}(\gamma) \subset \Omega$. Then*

$$\int_\gamma f(z)dz = 0.$$

PROOF. Set $\gamma_0 = \gamma$ and γ_1^k, $k = 1, 2, 3, 4$ be triangles obtained by breaking up γ_0 at the midpoints of the edges. Then

$$\int_\gamma f(z)dz = \sum_{k=1}^{4} \int_{\gamma_1^k} f(z)dz.$$

Thus, at least one of them satisfies

$$\left| \int_\gamma f(z)dz \right| \leq 4 \left| \int_{\gamma_1^k} f(z)dz \right|.$$

Choose this as γ_1. Iterating this process we obtain a sequence of triangles such that

$$\begin{aligned}
&\left| \int_\gamma f(z)dz \right| \leq 4^n \left| \int_{\gamma_n} f(z)dz \right| \\
&\text{ins}(\gamma_{n+1}) \subset \text{ins}(\gamma_n), \\
&2^n \ell(\gamma_{n+1}) = \ell(\gamma), \\
&2^n \operatorname{diam}(\gamma_{n+1}) = \operatorname{diam}(\gamma).
\end{aligned}$$

Since $\text{ins}(\gamma_n)$ are closed and $\operatorname{diam}(\gamma_n) \to 0$, we have

$$\{z_0\} = \bigcap_{n \geq 0} \text{ins}(\gamma_n).$$

We will use that f is holomorphic in z_0. Therefore, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z), \quad r = o(|z - z_0|).$$

Since the affine part has a primative, we have

$$\int_{\gamma_n} f(z)dz = \int_{\gamma_n} r(z)dz.$$

Fix $\epsilon > 0$. Then there is $\delta > 0$ such that for any $z \in B_\delta(z_0) \subset \Omega$, we have

$$|r(z)| \leq \epsilon |z - z_0|.$$

For n sufficiently large, $\text{ins}(\gamma_n) \subset B_\delta(z_0)$. Moreover, for any $z \in \operatorname{ran}\gamma_n$, $|z - z_0| \leq \ell(\gamma_n)$. Thus,

$$\left| \int_{\gamma_n} f(z)dz \right| \leq \epsilon \ell(\gamma_n)^2 = \epsilon \frac{1}{4^n} \ell(\gamma)^2.$$

Thus,

$$\left| \int_{\gamma} f(z)dz \right| \leq \epsilon \ell(\gamma)^2.$$

Since ϵ was arbitrary, this finishes the proof. □

This lemma will be needed to prove Theorem 2.3.

LEMMA A.2. *Let $r > 0$ and $w \in B_r(z_0)$. Then*

$$\int_{|z-z_0|=r} \frac{1}{z-w} dz = 2\pi i.$$

PROOF. Follows by direct computation. □

PROOF OF THEOREM 2.3. We find $R > r$ such that $\overline{B_r(z_0)} \subset B_R(z_0) \subset \Omega$ and thus we can assume that $\Omega = B_R(z_0)$ which is convex. Define

$$g_w(z) = \begin{cases} \frac{f(z)-f(w)}{z-w}, & z \neq w, \\ f'(w), & z = w. \end{cases}$$

Then g_w is continuous in Ω and holomorphic in $\Omega \setminus \{w\}$. Since Ω is convex, we obtain by Theorem 2.2

$$0 = \int_{|z-z_0|=r} \frac{f(z)-f(w)}{z-w} dz = \int_{|z-z_0|=r} \frac{f(z)}{z-w} dz - f(w) \int_{|z-z_0|=r} \frac{1}{z-w} dz.$$

The rest follows from Lemma A.2. □

PROOF OF THEOREM 2.4. By the affine transformation $z = z_0 + R\zeta$ we assume without loss of generality that $z_0 = 0$ and $R = 1$. The idea of proof is based on writing for $|z| = 1$ and $|w| < 1$

$$\frac{1}{z-w} = \frac{1}{z} \frac{1}{1 - wz^{-1}} = \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}}.$$

Plugging this into Cauchy's integral formula yields

$$f(w) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-w} dz = \frac{1}{2\pi i} \int_{|z|=1} \sum_{n=0}^{\infty} \frac{f(z) w^n}{z^{n+1}} dz$$

The result would follow if we were allowed to interchange integrals and sums. Instead we write for $N \in \mathbb{N}$

$$\frac{1}{z-w} = \sum_{n=0}^{N-1} \frac{w^n}{z^{n+1}} + \frac{w^N}{z^N} \frac{1}{z-w}.$$

Thus,

$$f(w) = \sum_{n=0}^{N-1} a_n w^n + R_N(w),$$

with

$$R_N(w) = w^N \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^N(z-w)} dz.$$

We have

$$|R_N(w)| \leq |w|^N \sup_{|z|=1} \frac{|f(z)|}{|z-w|}.$$

On $B_1(0)$, $\frac{|f(z)|}{|z-w|}$ is continuous and thus for any $\epsilon < 1$, $R_N(w) \to 0$ uniformly on $B_{1-\epsilon}(0)$. $\square$